Gaussian beams and ABCD

Gaussian beam solution to EM wave equation

Complex q formulation

“Standard” form of Gaussian beam equations

High-order modes

Gaussian beams and ABCD

- examples of Gaussian beam propagation

Gaussian beam solution to wave equation

- Use Fresnel integral to propagate a Gaussian beam

$$u(x,y,z)=\frac{i}{\lambda L}\mathrm{e}^{-ik\frac{x^2+y^2}{2L}}\iint e^{-\frac{x'^2+y'^2}{w^2}}\,\mathrm{e}^{-ik\frac{x'^2+y'^2}{2L}}\,\mathrm{e}^{+i\left(\beta_x x'+\beta_y y'\right)}\,dx'dy'$$

  - Combine quadratic terms in exponent:

$$\left(\frac{1}{w^2}+i\frac{k}{2L}\right)=i\frac{k}{2}\left(\frac{1}{L}-i\frac{2}{kw^2}\right)=i\frac{k}{2q}$$

  - Now integral is a F.T. of a complex Gaussian=Gaussian

$$u(x,y,z)=\frac{i}{\lambda L}\mathrm{e}^{-ik\frac{x^2+y^2}{2L}}\iint \mathrm{e}^{-ik\frac{x'^2+y'^2}{2q}}\,\mathrm{e}^{+i\left(\beta_x x'+\beta_y y'\right)}\,dx'\,dy'$$

$$\rightarrow u(r,z)=\frac{1}{q(z)}e^{-ik\frac{r^2}{2q(z)}} \qquad \frac{1}{q(z)}=\frac{1}{R(z)}-i\frac{\lambda}{\pi w^2(z)}$$

Complex q form for Gaussian beam

- This combines beam size and radius of curvature into one complex parameter
  - This form is used for for ABCD calculations

$$A(r,z)=A_0\frac{1}{1+i\xi}e^{-\frac{r^2}{w_0^2(1+i\xi)}} \qquad \to A(r,z)=\frac{1}{q(z)}e^{-ik\frac{r^2}{2q(z)}}$$

$$\frac{1}{q(z)}=\frac{1}{R(z)}-i\frac{\lambda}{\pi w^2(z)}$$

$$\frac{1}{q(z)}=\frac{1}{z+iz_R}=\frac{z}{z^2+z_R^2}-i\frac{z_R}{z^2+z_R^2}$$

$$=\frac{1}{R(z)}-i\frac{w_0^2}{z_R w^2(z)}$$

$$=\frac{1}{R(z)}-i\frac{\lambda}{\pi w^2(z)}=\frac{1}{R(z)}-i\frac{1}{Z(z)}$$

$$\frac{1}{R(z)}=\frac{1}{z\left(1+\frac{z_R^2}{z^2}\right)}=\frac{z}{z^2+z_R^2}$$

$$\frac{1}{w^2(z)}=\frac{1}{w_0^2\left(1+\frac{z^2}{z_R^2}\right)}=\frac{z_R^2}{w_0^2\left(z^2+z_R^2\right)}$$

Complex q vs standard form

$$u(r,z)=\frac{1}{q(z)}e^{-ik\frac{r^2}{2q(z)}} \qquad \text{with} \qquad \frac{1}{q(z)}=\frac{1}{R(z)}-i\frac{\lambda}{\pi w^2(z)}$$

Expand exponential:

$$\exp\left[-ik\frac{r^2}{2q(z)}\right]=\exp\left[-ik\frac{r^2}{2}\left(\frac{1}{R(z)}-i\frac{\lambda}{\pi w^2(z)}\right)\right]$$

$$=\exp\left[-ik\frac{r^2}{2}\frac{1}{R(z)}-i\frac{2\pi}{\lambda}\frac{r^2}{2}\left(-i\frac{\lambda}{\pi w^2(z)}\right)\right]=e^{-ik\frac{r^2}{2R(z)}}e^{-\frac{r^2}{w^2(z)}}$$

$$a+ib=\sqrt{a^2+b^2}\,e^{i\arctan(b/a)}$$

Expand leading inverse q:

$$\frac{1}{q(z)}=-i\left(\frac{iz}{z^2+z_R^2}+\frac{z_R}{z^2+z_R^2}\right)=-i\left(\frac{\sqrt{z^2+z_R^2}}{z^2+z_R^2}\right)e^{i\arctan(z/z_R)}$$

$$=-i\left(\frac{1}{z_R\sqrt{1+z^2/z_R^2}}\right)e^{i\arctan(z/z_R)}=\frac{w_0}{iz_Rw(z)}e^{i\eta(z)}$$

Standard form of Gaussian beam solutions

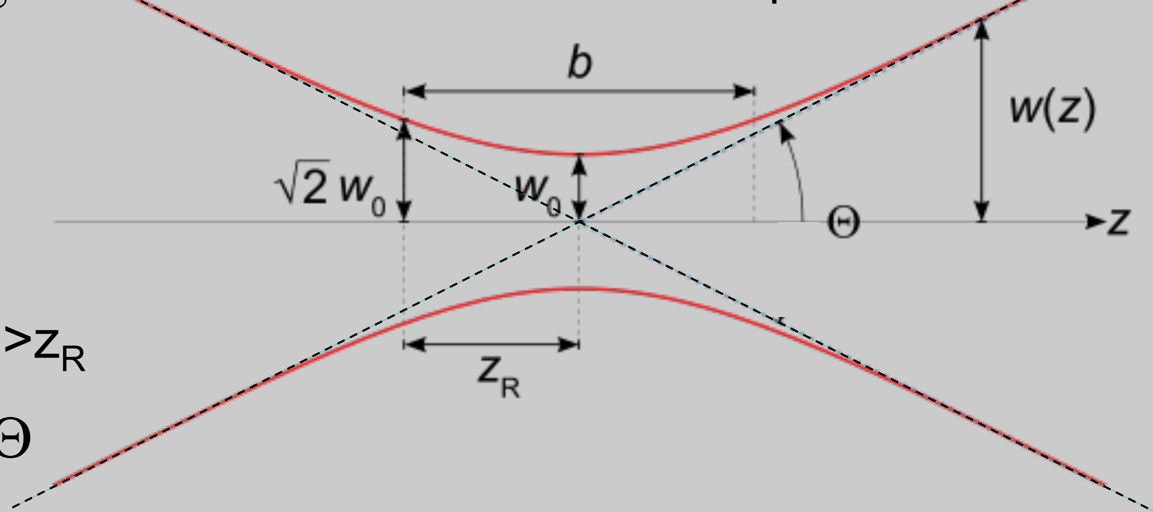
$$E(r,z,t) = A_0 e^{-i(kz-\omega t)} \frac{w_0}{w(z)} e^{-\frac{r^2}{w^2(z)}} e^{-i\frac{kr^2}{2R(z)}} e^{i\eta(z)}$$

Beam maintains a Gaussian profile as it propagates
- beam radius that varies with z
- Origin of z coordinate is at the beam waist
- Rayleigh length z_R defines collimation distance from focal plane

$$z_R = \frac{\pi w_0^2}{\lambda / n} = \frac{\text{beam area}}{\text{wavelength in medium}}$$

$$w(z) = w_0 \sqrt{1 + \frac{z^2}{z_R{}^2}}$$

b = confocal parameter
= depth of focus



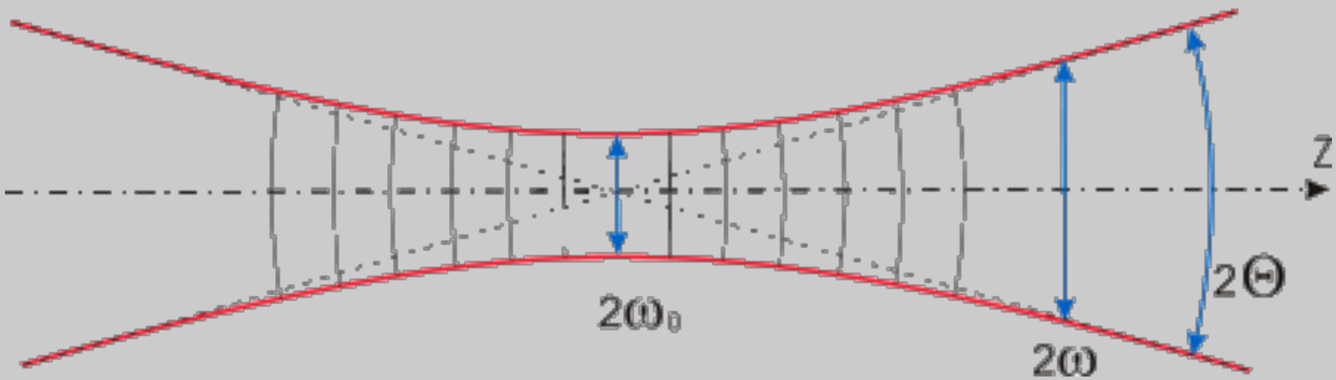
Geometric limit for $z \gg z_R$

$$w(z) = z\frac{w_0}{z_R} = z \tan\Theta$$

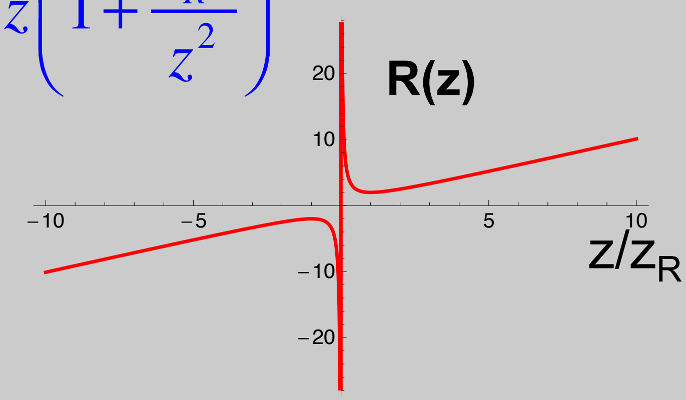
Evolution of wavefronts

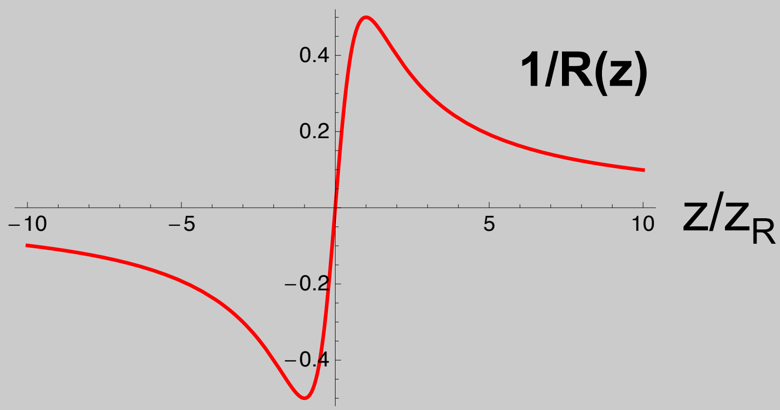
$$E(r,z,t) = A_0 e^{-i(kz-\omega t)} \frac{w_0}{w(z)} e^{-\frac{r^2}{w^2(z)}} e^{-i\frac{kr^2}{2R(z)}} e^{i\eta(z)}$$

- Wavefront curvature evolves with z as beam size changes



$$R(z) = z\left(1 + \frac{z_R^{\ 2}}{z^2}\right)$$





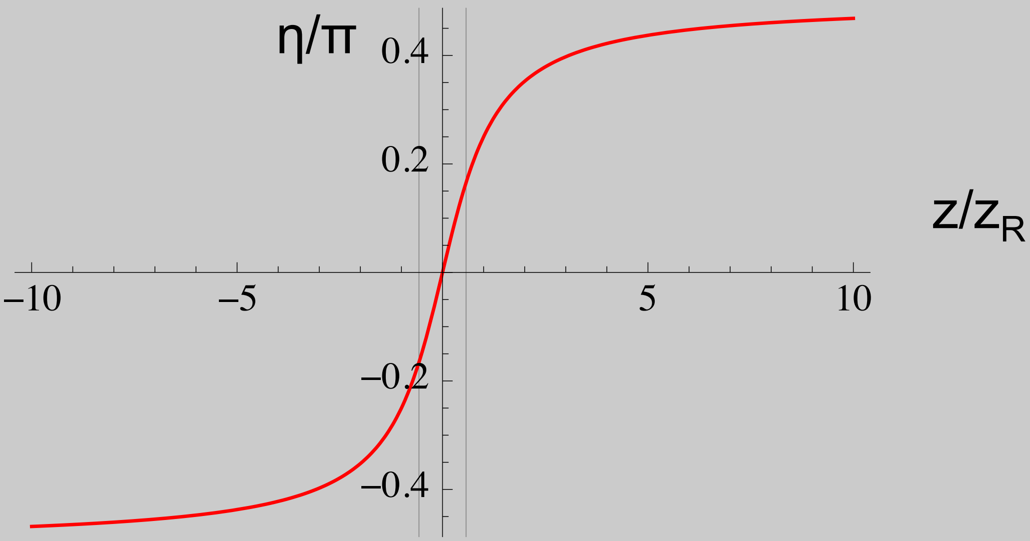
On-axis phase: Gouy phase

$$E(r,z,t) = A_0 e^{-i(kz-\eta(z)-\omega t)} \frac{w_0}{w(z)} e^{-\frac{r^2}{w^2(z)}} e^{-i\frac{kr^2}{2R(z)}}$$

- Because the wavefront changes from focusing to defocusing, on-axis phase advances with z

Gouy phase

$$\eta(z) = \arctan\left(\frac{z}{z_R}\right)$$



Higher-order Hermite-Gauss modes

– The Gaussian beam is just the lowest order mode solution to the wave equation

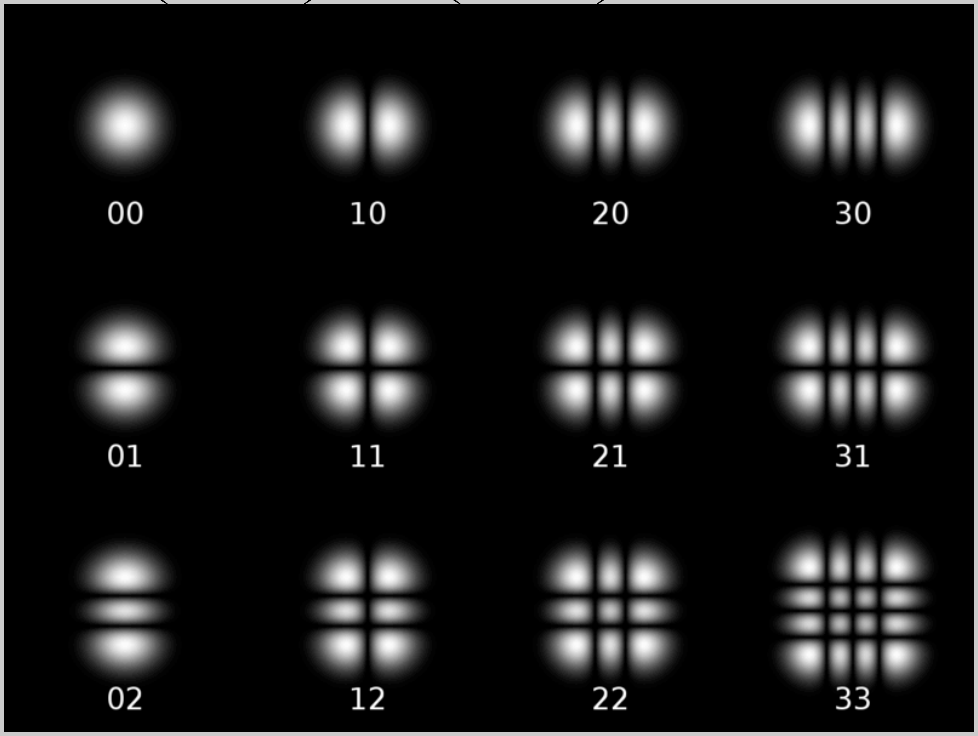
– x, y coordinates: Hermite-Gaussian modes

$$E(x,y,z) = A_0 e^{-i(kz-\eta_{lm}(z))} \frac{w_0}{w(z)} e^{-\frac{x^2+y^2}{w^2(z)}} H_l\left(\frac{\sqrt{2}x}{w(z)}\right) H_m\left(\frac{\sqrt{2}y}{w(z)}\right) e^{-i\frac{k(x^2+y^2)}{2R(z)}}$$

$$\eta_{lm} = (l+m+1)\tan^{-1}\left(\frac{z}{z_R}\right)$$

Transverse profile is maintained during propagation (scaled with w(z))

Hermite-Gauss functions are the same as solutions to quantum SHO

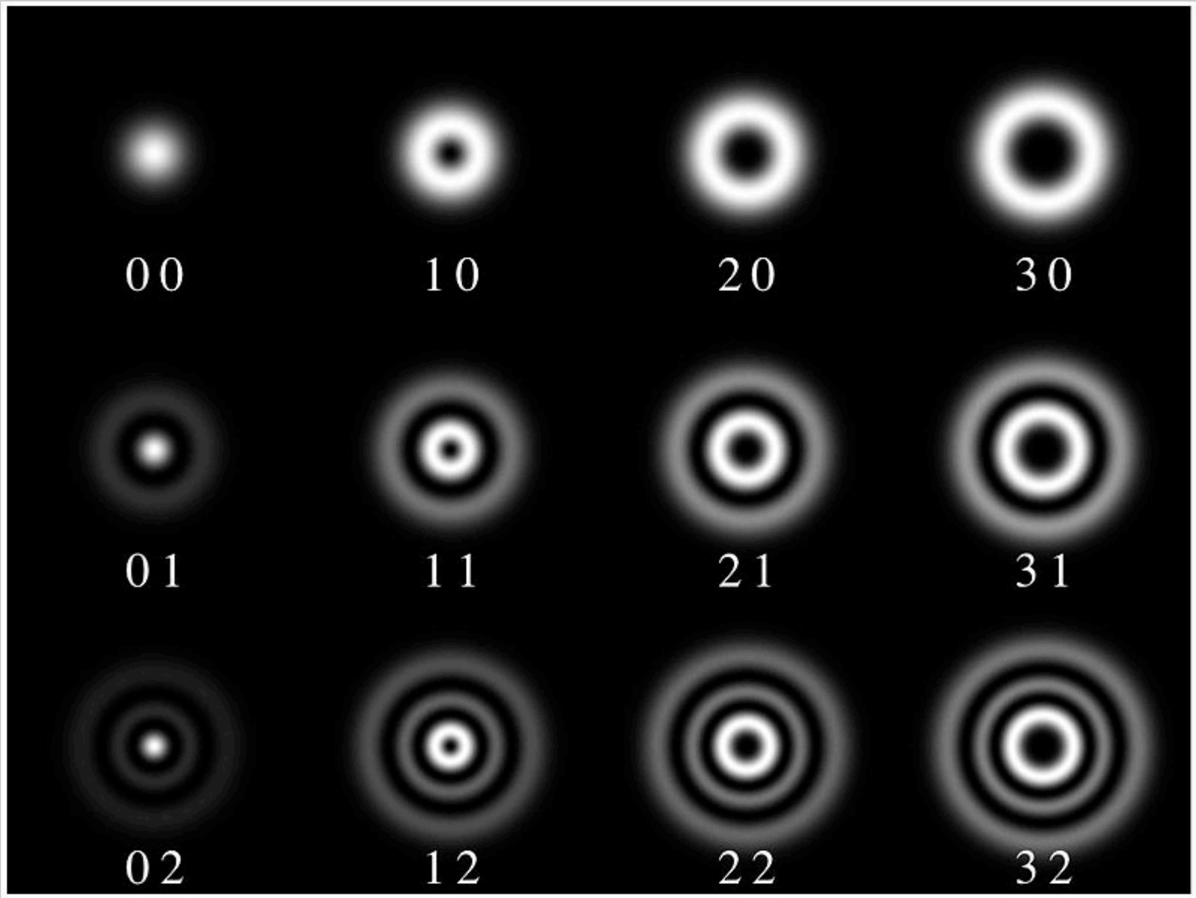


Higher-order LaGuerre-Gauss modes

– In cylindrical coordinates, alternate representation

– Azimuthal phase $\exp[im\phi]$ "vortex" phase

Example:
LG10 mode is a linear combination of HG10 and HG01



Difference between Siegman's complex q and standard form

$$u(r,z) = \frac{1}{q(z)} e^{-ik\frac{r^2}{2q(z)}} = \frac{1}{i z_R} \frac{w_0}{w(z)} e^{i\eta(z)} e^{-ik\frac{r^2}{2R(z)}} e^{-\frac{r^2}{w^2(z)}}$$

$$E(r,z,t) = A_0 \frac{w_0}{w(z)} e^{i(kz-\omega t)} e^{-\frac{r^2}{w^2(z)}} e^{i\frac{kr^2}{2R(z)}} e^{-i\eta(z)}$$

- Siegman's form for the complex q is used almost everywhere for the ABCD calculations.
- He uses the exp[+ I w t] convention, which accounts for the sign difference in the complex exponentials.
- With exp[-I w t] convention, define q as:

$$\frac{1}{q(z)} = \frac{1}{R(z)} + i \frac{\lambda}{\pi w^2(z)} = \frac{1}{z - i z_R}$$

Compare Boyd's form to standard:

- Boyd's complex form is consistent with standard Gaussian beam form

$$A(r,z) = A_0 \frac{1}{1+i\xi} e^{-\frac{r^2}{w_0^2(1+i\xi)}} = A_0 \frac{1}{1+iz/z_R} e^{-\frac{r^2}{w_0^2(1+iz/z_R)}}$$

$$\frac{1}{1+i\xi} = \frac{1}{1+iz/z_R} = \frac{z_R}{z_R+iz} = \frac{-iz_R}{z-iz_R} = \frac{-iz_R}{q(z)}$$

$$A(r,z) = A_0(-iz_R)\frac{1}{q(z)} e^{+\frac{iz_R r^2}{w_0^2 q(z)}} = -iz_R A_0 \frac{1}{q(z)} e^{+\frac{ikr^2}{2q(z)}}$$

Gaussian beams and ABCD

- General expression

$$q_1 = \frac{Aq_0 + B}{Cq_0 + D} \qquad \frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}$$

  - Since q is defined through its inverse, alternate:

$$q_1^{-1} = \frac{C + Dq_0^{-1}}{A + Bq_0^{-1}}$$

  - Note that ABCD matrices are the same as for raytrace
  - To use the matrices, we do **not** multiply matrix.vector
  - See Siegman, *Lasers* for proof

Simple Gaussian ABCD examples: translation

- translation

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)} = \frac{1}{z\left(1+\frac{z_R{}^2}{z^2}\right)} - i\frac{\lambda}{\pi w_0{}^2\left(1+\frac{z^2}{z_R{}^2}\right)}$$

$$= \frac{1}{z_R\left(1+\frac{z^2}{z_R{}^2}\right)}\left(z/z_R - i\right)$$

$$R(z) = z\left(1+\frac{z_R{}^2}{z^2}\right)$$

$$w(z) = w_0\sqrt{1+\frac{z^2}{z_R{}^2}}$$

$$q(z) = z_R\left(1+\frac{z^2}{z_R{}^2}\right)\frac{1}{\left(z/z_R - i\right)}$$

$$= z_R\left(1+\frac{z^2}{z_R{}^2}\right)\frac{z/z_R + i}{\left(1+\frac{z^2}{z_R{}^2}\right)} = z + iz_R$$

$$q(z) = z + iz_R \rightarrow q_1 = z_0 + L + iz_R$$

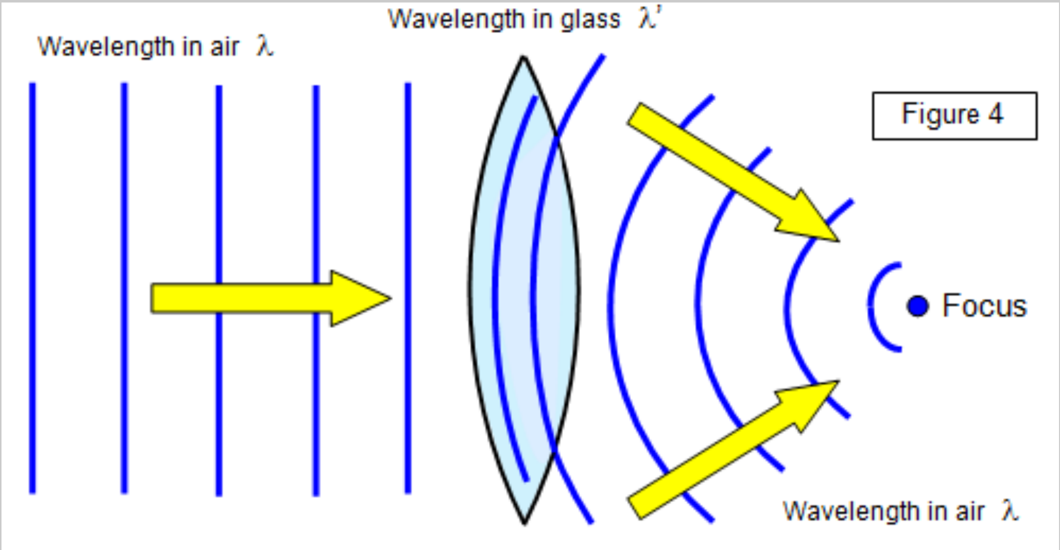
ABCD for translation:

$$\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$q_1 = \frac{Aq_0 + B}{Cq_0 + D} = q_0 + L$$

Simple Gaussian ABCD examples: lens

- Focusing by a lens
  - Radius of curvature is modified by lens: $\frac{1}{R'} = \frac{1}{R} - \frac{1}{f}$



Focusing by lens induces a negative ROC

$$\frac{1}{q_1} = \frac{1}{R} - \frac{1}{f} - i\frac{\lambda}{\pi w^2} = \frac{1}{q_0} - \frac{1}{f}$$

$$q_1^{-1} = \frac{C + Dq_0^{-1}}{A + Bq_0^{-1}}$$

$$\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

Focusing a Gaussian beam by a lens

- For a beam waist at lens entrance, distance from lens to focused waist is not exactly = f
- Define variables:

  w_{01} (w_{02}) = input (focused) beam waist radius

  z_{R1} (z_{R2}) = Rayleigh range for input (focused) beam

  z_m = distance from lens to focused beam waist

- Use Gaussian beam equations to propagate back to lens

$$w_{01} = w(z = -z_m) = w_{02}\sqrt{1 + \frac{z_m^2}{z_{R2}^2}} \qquad \rightarrow z_{R1} = \frac{\pi w_{01}^2}{\lambda} = z_{R2}\left(1 + \frac{z_m^2}{z_{R2}^2}\right)$$

$$R(z = -z_m) = -f = -z_m\left(1 + \frac{z_{R2}^2}{z_m^2}\right)$$

- Divide equations:

$$\frac{f}{z_{R1}} = \frac{z_m\left(1 + \frac{z_{R2}^2}{z_m^2}\right)}{z_{R2}\left(1 + \frac{z_m^2}{z_{R2}^2}\right)}$$

Gaussian beam focusing

- From prev slide:

$$\frac{f}{z_{R1}} = \frac{z_m\left(1+\frac{z_{R2}^2}{z_m^2}\right)}{z_{R2}\left(1+\frac{z_m^2}{z_{R2}^2}\right)}$$

- Let $x_1 = \frac{f}{z_{R1}}$ $\quad x_2 = \frac{z_{R2}}{z_m}$

$$\rightarrow x_1 = \frac{1}{x_2}\frac{1+x_2^2}{1+\frac{1}{x_2^2}} = \frac{1}{x_2}x_2^2\frac{1+x_2^2}{1+x_2^2} = x_2 \qquad \frac{f}{z_{R1}} = \frac{z_{R2}}{z_m}$$

- Go back to expressions for z_{R1} and *f*:

$$z_{R1} = z_{R2}\left(1+\frac{z_m^2}{z_{R2}^2}\right) = z_{R2}\left(1+\frac{z_{R1}^2}{f^2}\right) \qquad \rightarrow z_{R2} = \frac{z_{R1}}{1+\frac{z_{R1}^2}{f^2}}$$

$$f = z_m\left(1+\frac{z_{R2}^2}{z_m^2}\right) = z_m\left(1+\frac{f^2}{z_{R1}^2}\right) \qquad \rightarrow z_m = \frac{f}{1+\frac{f^2}{z_{R1}^2}}$$

Key parameter is the ratio: $\frac{f}{z_{R1}}$

Interpretation of results

- Ratio f/z_{R1} determines effect of input beam Rayleigh range on position of focus
  - Distance to beam waist is *shorter* than *f*
  - This matters only when z_{R1} is comparable to *f*

$$z_m = \frac{f}{1+\frac{f^2}{z_{R1}{}^2}}$$

- Focused spot size:
  - For $f/z_{R1} \ll 1$,

$$z_{R2} = \frac{z_{R1}}{1+\frac{z_{R1}{}^2}{f^2}} \rightarrow w_{02} = \frac{w_{01}}{\sqrt{1+\frac{z_{R1}{}^2}{f^2}}}$$

$$w_{02} = \frac{w_{01}}{\frac{z_{R1}}{f}\sqrt{1+\frac{f^2}{z_{R1}{}^2}}} \approx \frac{f\,w_{01}}{z_{R1}} = \frac{\lambda f}{\pi w_{01}}$$

$$w_{02} = \frac{2}{\pi}\lambda\frac{f}{2w_{01}} = \frac{2}{\pi}\lambda F/\#$$

F-number of focusing for a beam is defined as f/beam dia

Focused Gaussian beam: ABCD version

- Define input q: $$\frac{1}{q_0}=\frac{1}{R_0}-i\frac{\lambda}{\pi w_{01}{}^2}=-i\frac{1}{z_{R1}}$$
- Lens: $\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \rightarrow$ q just after lens: $$q_1{}^{-1}=\frac{C+Dq_0{}^{-1}}{A+Bq_0{}^{-1}}$$
$$\frac{1}{q_1}=\frac{1}{q_0}-\frac{1}{f}=-\frac{1}{f}-i\frac{1}{z_{R1}}$$
- Translation by z_m:
$$\begin{pmatrix} 1 & z_m \\ 0 & 1 \end{pmatrix} \rightarrow \qquad q_2{}^{-1}=\frac{C+Dq_1{}^{-1}}{A+Bq_1{}^{-1}}=\frac{-\frac{1}{f}-i\frac{1}{z_{R1}}}{1+z_m\left(-\frac{1}{f}-i\frac{1}{z_{R1}}\right)}$$
- At z_m, beam is at a waist, so
$$\frac{1}{q_2}=-i\frac{\lambda}{\pi w_{01}{}^2}=-i\frac{1}{z_{R2}}$$

Then solve by setting Re[ ] = 0 above.