2_1_08

Note Title 1/31/2008

$$\Delta p \, \Delta x \geq \frac{\hbar}{2}$$

$\sigma_p = \Delta x$ σ is more common in stastics. Remember σ_p is the standard deviation

Lower Limit $\Delta x \, \Delta p = \hbar/2$

For QHO, total energy

$$E = \frac{p^2}{2m} + \frac{1}{2} m w^2 x^2$$

must, therefore, be at least

$$\frac{(\Delta p)^2}{2m} + \frac{1}{2} m w^2 (\Delta x)^2$$

$\Delta p =$ momentum uncertainty
$\Delta x =$ position uncertainty

Usin $\Delta p \, \Delta x = \hbar/2$ we have

$$E = \frac{1}{2m} \left(\frac{\hbar}{2 \Delta x} \right)^2 + \frac{1}{2} m w^2 \Delta x^2$$

$$\frac{\hbar^2}{8m} (\Delta x)^{-2} + \frac{1}{2} m w^2 \Delta x^2$$

minimize this w.r.t. Δx

$$\frac{dE}{d(\Delta x)} \to 0$$

$$-\frac{\hbar}{4m}(\Delta x)^{-3} + m\omega^2 \Delta x = 0$$

$$\frac{\hbar^2}{4m}\frac{1}{(\Delta x)^3} = m\omega^2 \Delta x$$

$$\Delta x^4 = \frac{\hbar^2}{4m^2\omega^2}$$

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega}}$$

insert this into

$$E = \frac{1}{2m}\left(\frac{\hbar}{2\Delta x}\right)^2 + \frac{1}{2}m\omega^2 \Delta x^2$$

to get

$$E_0 = \frac{\hbar^{\cancel{2}\,1}}{8\cancel{m}}\frac{2\cancel{m}\omega}{\cancel{\hbar}} + \frac{1}{2}\cancel{m}\omega^{\cancel{2}\,1}\frac{\hbar}{2\cancel{m}\cancel{\omega}}$$

$$= \frac{\hbar\omega}{4} + \frac{\hbar\omega}{4} = \frac{\hbar\omega}{2}$$

This agrees with what we got before using a and a^+.

The energy of a system governed by H.O. cannot be zero. So-called zero point vibration energy

E.g. specific heat of ideal gas (Einstein/Planck)

$$\epsilon = \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1} + \frac{\hbar\omega}{2}$$

as $T \to 0$ $\quad \epsilon \to \frac{\hbar\omega}{2}$

E.g. Keeps Helium 4 from solidifying as $T \to 0$.

E.g. Casimir Effect

$$\psi(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)$$

$$\langle p \rangle = \frac{2}{a}\int_0^a \sin\left(\frac{\pi x}{a}\right)\left(-i\hbar\frac{d}{dx}\right)\left(\sin\left(\frac{\pi x}{a}\right)\right)dx$$

$$= -i\hbar\,\frac{2}{a}\,\frac{\pi}{a}\underbrace{\int_0^a \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right)dx}_{=0}$$

$$\langle p^2 \rangle = +\hbar^2\left(\frac{\pi}{a}\right)^2 \underbrace{\frac{2}{a}\int_0^a \sin^2\left(\frac{\pi x}{a}\right)dx}_{=1}$$

$$\sigma_p^2 = \langle p^2 \rangle - \langle p \rangle^2 = \hbar^2\left(\frac{\pi}{a}\right)^2$$

$$\langle x \rangle = \frac{2}{a}\int_0^a x\,\sin^2\left(\frac{\pi x}{a}\right)dx$$

$$\frac{\pi x}{a} = y \qquad dx = \frac{a}{\pi}dy$$

$$\frac{2}{a}\left(\frac{a}{\pi}\right)^2\int_0^\pi y\,\sin^2(y)\,dy$$

$$\frac{2}{a}\left(\frac{a}{\pi}\right)^2\left[\frac{y^2}{4} - \frac{1}{4}y\sin(2y) - \frac{1}{8}\cos(2y)\right]_0^\pi$$

$$\frac{2}{a}\left(\frac{a}{\pi}\right)^2\left[\left\{\frac{\pi^2}{4} - \frac{1}{8}\right\} + \frac{1}{8}\right] = \frac{2}{a}\,\frac{a^2}{4} = \frac{a}{2}$$

$$\langle x^2 \rangle = \frac{2}{a} \int_0^a x^2 \sin^2\left(\frac{\pi x}{a}\right) dx$$

$$= \frac{1}{6} a^2 \left(2 - \frac{3}{\pi^2}\right)$$

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$= \frac{1}{6} a^2 \left(2 - 3/\pi^2\right) - \frac{a^2}{2^2}$$

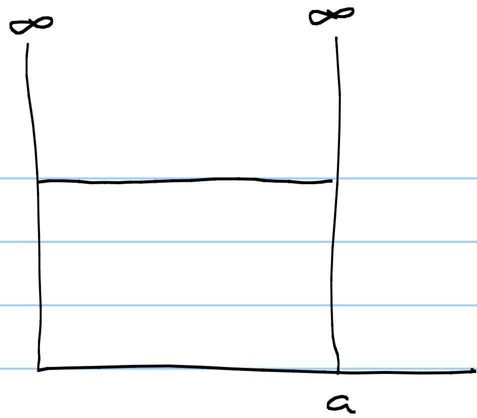
$$= a^2 \left[\frac{1}{3} - \frac{1}{4} - \frac{1}{2\pi^2}\right]$$

$$= a^2 \left(\frac{1}{12} - \frac{1}{2\pi^2}\right)$$

$$\sigma_x \sigma_p = \hbar \left(\frac{\pi}{a}\right) a \sqrt{\left(\frac{1}{12} - \frac{1}{2\pi^2}\right)}$$

$$= \hbar \sqrt{\frac{\pi^2 - 6}{12}} \approx \hbar \, 1.4$$

So certainly $\sigma_x \sigma_p > \frac{\hbar}{2}$



$$\psi(x,0) = \begin{cases} \frac{1}{\sqrt{a}} & 0 \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

$$\sqrt{\frac{1}{a}} = \sum_{N=1}^{\infty} c_1 \sqrt{\frac{2}{a}} \sin\left(\frac{N\pi x}{a}\right)$$

$$c_N = \sqrt{\frac{1}{a}}\sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{N\pi x}{a}\right) dx$$

$$= -\sqrt{\frac{1}{a}}\sqrt{\frac{2}{a}} \left[\frac{a}{N\pi} \cos\left(\frac{N\pi x}{a}\right)\right]_0^a$$

$$= -\sqrt{2}\,\frac{1}{N\pi}\left[\cos(N\pi) - 1\right]$$

$$c_N = \begin{cases} \sqrt{2}\,\frac{2}{\pi}\,\frac{1}{N} & N \text{ odd} \\ 0 & N \text{ even} \end{cases}$$

NB

$$\sum_{N=1}^{\infty} |c_N|^2 = \frac{8}{\pi^2} \sum_{N=1,3,\ldots} \frac{1}{N^2}$$

$$= \frac{8}{\pi^2}\left[1 + \frac{1}{9} + \frac{1}{25} + \cdots\right] = 1$$

proof $$\sum_{N=1}^{\infty} \frac{1}{N^2} = \zeta(2) = \frac{\pi^2}{6}$$

$$S \equiv 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} \cdots$$

$$= \underbrace{\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right)}_{\zeta(2)} - \underbrace{\left(\frac{1}{4} + \frac{1}{16} + \frac{1}{36} \cdots\right)}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= \frac{1}{4} \zeta(2)$$

$$S = \zeta(2) - \frac{1}{4}\zeta(2) = \frac{3}{4}\zeta(2)$$

$$= \frac{3}{4}\frac{\pi^2}{6} = \frac{\pi^2}{8}$$

Back to the story

initial state uniformly distributed on $[0, a]$

$$\Psi(x,0) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$c_n = \begin{cases} \frac{2}{\pi}\frac{1}{n} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$