Note Title 10/4/2006

10/4/06

Exam Review Friday.

Practice exam in lieu of Homework.

10/4/06

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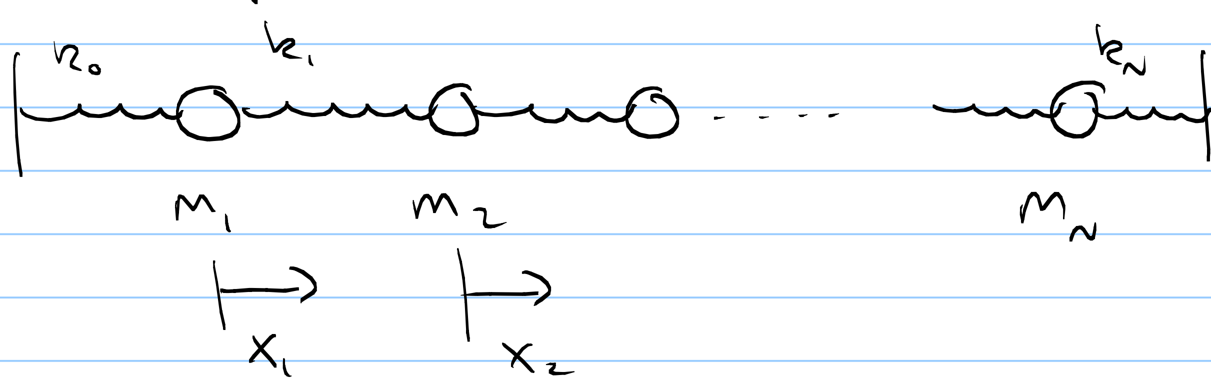
Symmetric quadratic forms

$(\vec{x}, A\vec{x})$ where A is symmetric: $A^T = A$

These arise frequently in physics

Example N masses on a lattice



For each mass $F = ma$

$$m_j \ddot{x}_j = k_{j+1}(x_{j+1} - x_j) - k_j(x_j - x_{j-1})$$

$$m_j \ddot{x}_j = \underbrace{k_{j+1} x_{j+1} - (k_{j+1} + k_j) x_j + k_j x_{j-1}}_{\text{one row of } A \cdot \vec{x}}$$

$$\begin{bmatrix} & & & \\ & & & \\ k_j & -(k_{j+1}+k_j) & k_{j+1} & \\ & & & \end{bmatrix} \begin{bmatrix} \vdots \\ x_{j-1} \\ x_j \\ x_{j+1} \\ \vdots \end{bmatrix}$$

special case: all the springs/masses are the same

$$k_i = k$$

$$m_i = m$$

$$m \begin{bmatrix} \ddot{x}_1 \\ \vdots \\ \ddot{x}_N \end{bmatrix} = k \underbrace{\begin{bmatrix} -2 & 1 & 0 & \cdots & \\ 1 & -2 & 1 & & \\ 0 & 1 & -2 & 1 & \\ & & & \ddots & \\ & & & 1 & -2 \end{bmatrix}}_{K \quad \text{matrix}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

$$\underbrace{}_{\ddot{\vec{u}}} \qquad \underbrace{}_{\vec{u}}$$

$$\boxed{m \ddot{\vec{u}} = K \vec{u}}$$

potential energy of the system is

$$\frac{1}{2} (\vec{x}, K \vec{x})$$ N-Dim Quadratic form

Kinetic Energy

$$\frac{1}{2} (\dot{\vec{x}}, M \dot{\vec{x}})$$

$$M = \begin{pmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_N \end{pmatrix}$$

real
Symmetric matrix $A \in \mathbb{R}^{n \times n}$

The quadratic form associated with the matrix is

$$(\vec{x}, A\vec{x})$$

There are many other examples

Inertia, Stress, ...

You're used to seeing quadratic forms like this:

$$x^2 + 6xy - 2y^2 - 2yz + z^2 = 24$$

[look at mathematica]

we want to write this as

$$(\vec{x}, A\vec{x})$$

$$= (x, y, z) \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

step 1

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{pmatrix}$$

step 2

$$(x, y, z) \begin{pmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{pmatrix}$$

$$x(a_{11}x + a_{12}y + a_{13}z) +$$

$$y(a_{21}x + a_{22}y + a_{23}z) +$$

$z(a_{31}x + a_{32}y + a_{33}z)$

$= a_{11}x^2 + \underline{a_{12}}\,xy + a_{13}\,xz$

$\underline{a_{21}}\,xy + a_{22}\,y^2 + a_{23}\,yz$

$a_{31}\,xz + a_{32}\,yz + a_{33}\,z^2$

we want $a_{31} = a_{13}$

$a_{21} = a_{12}$

$a_{32} = a_{23}$

$x^2 + 6xy - 2y^2 - 2yz + z^2 = 24$

$a_{11} = 1 \qquad a_{33} = 1$

$a_{12} = a_{21} = 3 \qquad a_{22} = -2$

$a_{32} = a_{23} = -1$

$$\begin{bmatrix} 1 & 3 & 0 \\ 3 & -2 & -2 \\ 0 & -1 & 1 \end{bmatrix} = A$$

$$(x, y, z)\begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= x^2 + 6xy - 2y^2 - 2yz + z^2$$

Now we can diagonalize A

characteristic polynomial =

$$\operatorname{Det}\begin{bmatrix} 1-\lambda & 3 & 0 \\ 3 & -2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix}$$

$$= -\lambda^3 + 13\lambda - 12$$

$$= -(\lambda - 1)(\lambda + 4)(\lambda - 3)$$

$$\lambda_1 = 1 \quad \lambda_2 = -4 \quad \lambda_3 = 3$$

So $A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & \\ & -4 & \\ & & 3 \end{pmatrix}$

$$(x', y', z') \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -3 \end{pmatrix} (x', y', z')$$

$$= x'^2 - 4y'^2 + 3z'^2 = 24$$

General theorem $A \in R^N$

Suppose A has N linearly independent $\mathcal{E}$-vectors.

Let S be the matrix formed from these $\mathcal{E}$-vectors then

$$S^{-1} A S = \Lambda$$

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_N)$

sketch

Proof

$$A \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} | & | & \\ \vec{s}_1 & \vec{s}_2 & \cdots \\ | & | & \end{bmatrix} =$$

$$A \cdot \vec{s}_1 = \lambda_1 \vec{s}_1 \quad \cdots \quad \text{etc.}$$

$$\begin{bmatrix} \lambda_1 \vec{s}_1 & \lambda_2 \vec{s}_2 & \cdots \end{bmatrix} = \begin{bmatrix} \vec{s}_1 & \vec{s}_2 & \cdots \end{bmatrix} \mathrm{diag}(\lambda_1 .. \lambda_n)$$

$$= S\Lambda$$

$$\Rightarrow \quad AS = S\Lambda$$

$$\Rightarrow \quad S^{-1} A S = S\Lambda$$

examples

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\text{Det}\begin{pmatrix} 0-\lambda & 1 \\ 0 & 0-\lambda \end{pmatrix} = \underline{\lambda^2 = 0}$$

So the E-vectors are elements of the null space.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \quad y=0$$

x unconstrained

so any vector of the form $\begin{pmatrix} x \\ 0 \end{pmatrix}$ is an E-vector

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} x \\ 0 \end{pmatrix} \quad \checkmark$$

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ex. $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ $\lambda_{1,2} = 3$

Hence Σ-vectors cannot be lin. independent

char. poly: $\mathrm{Det}\begin{pmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix} = 0$

$$(3-\lambda)^2 = 0 \qquad \lambda_{1,2} = 3$$

Theorem

If n eigenvectors of an $n \times n$ matrix correspond to different eigenvalues, then the eigenvectors are lin. independent.

Theorem if $A = A^T$ $A \in \mathbb{R}^{n \times n}$

Then A can be factored as $A = Q \Lambda Q^T$ with orthonormal ε-vectors and real ε-values

Read the book

skip sec. 13 on Groups

Read sec 14 on vector
spaces

Last 2 lectures have
followed sec. 11 & 12

1-D Gaussian e^{-x^2}

N-D Gaussian $e^{-(x, Ax)}$

$e^{-x^2-y^2}$ versus $e^{-x^2-2xy-y^2}$

$\Downarrow$

$e^{-x^2} e^{-y^2}$