MATH 225 - Differential Equations June 1 , 2008

Homework 5, Field 2008 **Not Due Date:** June 5, 2008

SECOND-ORDER LINEAR EQUATIONS - MASS-SPRING SYSTEMS - POWER SERIES

1. Consider the following second-order linear ordinary differential equation with constant coefficients,

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t), \quad a, b, c \in \mathbb{R}. \tag{1}$$

Solve (1) for the following cases, when possible solve for any unknown coefficients,

(a) $a = 1$, $b = -2$, $c = -3$, $f(t) = 3e^{-t}$.

(b) $a = 1$, $b = 4$, $c = 4$, $f(t) = 3e^{-t} + t^2$.

(c) $a = 1$, $b = -4$, $c = 13$, $f(t) = 0$, subject to, $y(0) = 1$ and $y'(0) = -1$.

(d) $a = 1$, $b = 0$, $c = 9$, $f(t) = 2\sin(2t)$.

(e) $a = 1$, $b = 0$, $c = 9$, $f(t) = \cos(3t)$.

2. Consider the model equation for a mass suspended from an ideal spring. If we include the effects of frictional forces and an external applied force, $f(t)$, we can derive from force laws[3] the second-order linear ordinary differential equations with constant coefficients:

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f(t), \quad m, b, k \in R^+ \cup \{0\}, \tag{2}$$

(a) Convert the second-order linear ODE (3) to a system of first-order ODE's.

(b) If $f(t) = 0$ for all t and $b = 0$ we call this unforced oscillator *simple*. Show that the fixed point of an unforced simple harmonic oscillator is always a center.[4]

(c) We now consider the effects of friction using MASSSPRING and the systems defined by $m = k = 1$ and $b_1 = 0$, $b_2 = 0.5$, $b_3 = 1$, $b_4 = 1.5$, $b_5 = 2$. For each of the previous systems plot a trajectory whose initial condition is somewhere near the center of the first quadrant and using these plots describe effects of friction on the long-term behavior to each of the trajectories.[5]

3. Now we consider the effects of external forcing on a simple harmonic oscillator.[6] Of all of the external forces to consider the most interesting involve periodic forcing. Here we consider an applied force given by $f(t) = F\cos(\omega t)$, $F, \omega \in \mathbb{R}^+ \cup \{0\}$. Run the program FORCEDMASSSPRING for all permutations of the values, $F_1 = 1$, $F_2 = 2$, $\omega_1 = 0$, $\omega_2 = 0.5$, $\omega_3 = 0.75$, $\omega_4 = 1$, plotting the trajectories whose initial condition is roughly in the center of the first quadrant. Using this information respond to the following:

(a) How does constant forcing effect the fixed point of the system?[7]

(b) Now considering the parameter ω, for $\omega < 1$, how does oscillatory forcing effect the behavior of trajectories in phase space?[8]

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[3]Remember that when deriving this equation we used Hook's law, which says that in the elastic limit the restoring force is linearly pro-
rtional to the displacement/deformation. Outside of this limit the relationship becomes nonlinear and can be used to explain phenomenon
e non-reversible deformations associated with large displacements.

[4]We may call this oscillator simple but it is also classic example of a conservative system. In this case it is energy, which is conserved.
ne notion of conserved quantities will be explored in the next homework and applied to nonlinear systems in chapter 5.3

[5]Friction is considered a dissipative effect. Normally when discussing a conservative system it is common to also discuss the effects of
rresponding dissipative effects. This may not always be as simple as studying the effects of a single term in the system.

[6]What we are about to see here is so important to physical systems prone to oscillations that we will study it again in the next homework
rough the model equation (3) and not the displacement-velocity system found in problem (2).

[7]In mathematical terms the time-independent inhomogeneity has shifted the fixed point to be off the origin.

[8]Since the system is no longer autonomous there are no *fixed points*, however the trajectories do appear to be 'orbiting' points in phase
ace and one of them seems to correspond to the fixed point of part (a). That is to say, though we do not have *fixed points*, by definition,
r understanding of them can be useful in describing non-autonomous cases.

(c) Consider the case where $\omega = 0.75$ and looking at the graph of y versus t notice that the curve is an oscillatory function whose amplitude is itself also oscillating.[9] This pattern, which occurs when the frequency of forcing nears the frequency of natural oscillation, is called a beat pattern. Using `http://en.wikipedia.org/wiki/Beat_\%28acoustics\%29` explain the connection between this mass-spring phenomenon and acoustics.

(d) Explain what occurs to the mass-spring system when $\omega = 1$ and give examples of other phenomenon, which have similar qualitative features. [10]

4. Consider the governing equation for a mass suspended from an ideal spring. Including forces due to friction, and an external applied force, $f(t)$, leads to the second order linear ordinary differential equations with constant coefficients:

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f(t), \quad m, b, k \in R^+ \cup \{0\}, \tag{3}$$

(a) If $b = 0$ then the oscillator is called *simple*. Show that from the homogeneous (not forced) simple harmonic oscillator one can derive the conservation law $E_{total} = \dfrac{mv^2}{2} + \dfrac{ky^2}{2}$ where $v = \dfrac{dy}{dt}$ and E_{total} is a constant.[1]

(b) Assume that $m = k = 2$ and graph the conservation law in the yv-plane for $E_{total} = 1, 4, 9$. [2]

(c) Show that, for an unforced simple harmonic oscillator, the that the solution can be written as $y_h(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$. Determine w_0 in terms of m and k.

(d) Let $f(t) = \cos(\alpha t)$, $\alpha \in \mathbb{R}$. Pick the form of the particular solution, $y_p(t)$, for the simple harmonic oscillator. What happens when $\alpha = w_0$? Write down the functional form of the general solution for both of these cases. (DO NOT SOLVE FOR THE UNDETERMINED COEFFICIENTS)

(e) Consider the program BEATSANDRESONANCE where $a = 1.5$.

i. Describe what happens to the general solution (green) as the circular frequency, ω, of forcing is changed from 0.5 through 1.5. [3]

ii. Describe the changes to the homogenous solution (blurple) and nonhomogenous solution (red), relative to one another, as the frequency of forcing is changed from 0.5 through 1.5.

iii. If the energy of a single cycle of a sinusoidal-wave is proportional to the square of the amplitude then compare the amount of energy in one beat envelope for when $\omega \approx 0.5$ to when $\omega \approx 1.2$. What happens to the energy when $\omega \approx 1.5$?

5. Consider the ordinary differential equation:

$$y'' - y = 0 \tag{4}$$

We know that the general solution to this equation is $y(t) = c_1e^t + c_2e^{-t}$. It is common to write the solutions to (4) in terms of the hyperbolic trigonometric functions, $\sinh(t) = \dfrac{e^t - e^{-t}}{2}$, $\cosh(t) = \dfrac{e^t + e^{-t}}{2}$.

(a) Show that $y(t) = b_1\sinh(t) + b_2\cosh(t)$ is a solution to the differential equation (4).

(b) Show that if $c_1 = \dfrac{b_1 + b_2}{2}$ and $c_2 = \dfrac{b_1 - b_2}{2}$ then $y(t) = c_1e^t + c_2e^{-t} = b_1\cosh(t) + b_2\sinh(t)$.

(c) Assume that $y(t) = \displaystyle\sum_{n=0}^{\infty} a_n t^n$ and find the general solution of (4) in terms of the hyperbolic sine and cosine functions. [4]

[9] We say that the higher frequency oscillations are bounded by a lower frequency *envelope*. Qualitative changes to this envelope are portant in the diffraction pattern of waves and as we will see, in a moment, resonance.

[10] You may want to consider the following website to guide your thoughts `http://en.wikipedia.org/wiki/Resonance`[11]

[1] In physics one would call this conservation law a *constant of motion*.

[2] These *constants of motion* are nothing more than trajectories of the simple harmonic oscillator in the phase-plane.

[3] You may find it useful to toggle the *Envelope* feature.

[4] The hyperbolic sine and cosine have the following Taylor's series representations centered about $t = 0$:

$$\cosh(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \qquad \sinh(t) = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \tag{5}$$

1.

a) Given,

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 3e^{-t}$$

Assuming $y(t) = e^{\lambda t} \Rightarrow y'' - 2y' - 3y = e^{\lambda t}(\lambda^2 - 2\lambda - 3) = 0$ gives the roots

$$\lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$$

$$\lambda_1 = 3 \quad , \quad \lambda_2 = -1$$

and homogeneous soln:

$$y_h(t) = C_1 e^{-t} + C_2 e^{3t}$$

Our particular soln is of the form

$$y_p(t) = Ae^{-t} \text{ where } y_p'' - 2y_p' - 3y_p = Ae^{-t} + 2Ae^{-t} - 3Ae^{-t} = $$
$$= 0 \quad (\text{Inconsistent})$$

Noting that $y_p \propto e^{-t}$ implies we should guess,

$$y_p(t) = Ate^{-t}$$

$$\Rightarrow \quad y_p'(t) = Ae^{-t} - Ate^{-t}$$

$$y_p''(t) = -Ae^{-t} - Ae^{-t} + Ate^{-t}$$

Thus,

$$y'' - 2y' - 3y = (-2Ae^{-t} + Ate^{-t}) - 2Ae^{-t} + 2Ate^{-t} - 3Ate^{-t} = 3e^{-t}$$

$$\Rightarrow -2A - 2A = 3 \Rightarrow A = -\frac{3}{4}$$

and the general soln is,

$$y(t) = C_1 e^{-t} + C_2 e^{3t} - \frac{3t}{4} e^{-t}$$

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b. Given

$$y'' + 4y' + 4y = 3e^{-t} + t^2$$

$$y_h(t) = C_1 e^{-2t} + C_2 t e^{-2t}$$

$$y_p(t) = Ae^{-t} + Bt^2 + Ct + D$$

$$\Rightarrow y'' + 4y' + 4y = Ae^{-t} + 2B + -4Ae^{-t} + 8Bt + 4C +$$

$$+ 4Ae^{-t} + 4Bt^2 + 4Ct + 4D = 3e^{-t} + t^2$$

Const: $2B + 4C + 4D = 0 \Rightarrow \frac{1}{2} + -2 + 4D = 0 \Rightarrow D = \frac{5}{16}$

$\Rightarrow$ $\cancel{D \neq 0}$ $\quad t: 4C + 8B \Rightarrow C = -\frac{1}{2}$

$t^2: 4B + 0 = 1 \Rightarrow B = \frac{1}{4}$

$e^{-t}: A - 4A + 4A = 3$

$A = 3$

$\Rightarrow$

$$y(t) = C_1 e^{-2t} + C_2 t e^{-2t} + 3e^{-t} + \frac{t^2}{4} - \frac{t}{2} + \frac{5}{16}$$

c) $y'' - 4y' + 13 = 0$, $y(t) = e^{\lambda t} \Rightarrow \lambda^2 - 4\lambda + 13 = 0$

$$\lambda = \frac{4 \pm \sqrt{16 - 4(13)}}{2} = 2 \pm 3i$$

$$\Rightarrow y(t) = C_1 e^{2t}\cos(3t) + C_2 e^{2t}\sin(3t)$$

$y(0) = 1 = C_1$

$y'(0) = -1 = 2C_1 + 3C_2 \Rightarrow C_2 = -1$

$$y(t) = e^{2t}\cos(3t) - e^{2t}\sin(3t)$$

d) $y'' + 9y = 2\sin(2t) \Rightarrow y_n(t) = C_1\cos(3t) + C_2\sin(3t)$

$y_p(t) = Ae^{2it} \Rightarrow y'' + 9y = -4Ae^{2it} + 9Ae^{2it} = 2e^{2it}$

$\Rightarrow A = \frac{2}{5}$

Since $\cancel{\ }\, f(t) = \cancel{Re}\, Img\{2e^{2it}\}$

we take $Img\{y_p(t)\} = \frac{2}{5}\sin(2t)$

$\Rightarrow y(t) = y_n(t) + \frac{2}{5}\sin(2t)$

e) $y'' + 9y = \cos(3t) \Rightarrow y_n(t)$ from above,

Note $f(t) \propto \cancel{y_n(t)}\,\cos(3t)$ thus,

$y_p(t) = Ate^{3it} \Rightarrow y'' + 9y = 6iAe^{3it} = e^{3it}$

$y_p'(t) = Ae^{3it} + 3iAte^{3it}$ $\qquad \Rightarrow A = \frac{-i}{6}$

$y_p''(t) = 3iAe^{3it} +$

$+ 3iAe^{3it} + -9Ate^{3it}$

$\Rightarrow y(t) = y_n(t) + Real\{\frac{-i}{6}te^{3it}\} = y_n(t) + \frac{1}{6}t\sin(3t)$

2)

a) $my'' + by' + ky = f(t)$

let $y' = v \Rightarrow y'' = v' = -\frac{b}{m}v - \frac{k}{m}y + \frac{f(t)}{m}$

$$\frac{d}{dt}\begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}\begin{bmatrix} y \\ v \end{bmatrix} + \begin{bmatrix} \cancel{f(t)}\ 0 \\ f(t)/m \end{bmatrix}$$

b) Assume $b=0, f(t)=0 \Rightarrow A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}$

$T=0$ $D = k/m \Rightarrow$ the system's fixed point is a center.

c) The introduction of friction $\Rightarrow T<0 \Rightarrow$
$\Rightarrow$ System is a sink. For large enough values the system will move from a spiral sink to a Real sink.

3) a) The system doesn't ~~really~~ have a fixed point at the origin. The constant forcing moves it off (0,0).

b) The periodic forcing oscillates the system about the fixed points associated with positive + neg Amp. of the periodic forcing.

c) In the study of sound beats appear when two sounds of similar freq. undergo constructive and destructive interference which produces a superposition wave which has a Rapid oscillations bounded by a slowly var. Envelope.

d) At $\omega = 1$ Resonance occurs which is Characterized by the unbounded increase in amplitude in time.
Search text or wikipedia for more Physical Examples.

#/.

2) a Let $\gamma = 0$ and let $g(t) \equiv 0$. Then (3) becomes,

$$my'' + ky = 0 \iff my''y' + kyy' = 0 \iff$$

$$\iff \int my''y'dt + k\int yy'dt = \int 0\,dt \iff$$

$$\iff m\int u_1 du_1 + k\int u_2 du_2 = C \quad , \quad u_1 = y' \quad u_2 = y$$

$$\iff \frac{mu_1^2}{2} + \frac{k}{2}u_2^2 = \frac{m(y')^2}{2} + \frac{k}{2}y^2 = E_{total}$$

b. For $my'' + ky = 0$ assume $y(t) = e^{rt}$ to get

$$my'' + ky = mr^2e^{rt} + ke^{rt} = 0 \Rightarrow r^2 = \frac{-k}{m} \Rightarrow r = \pm\sqrt{\frac{k}{m}}\,i = \pm w_0 i$$

since $k, m \in \mathbb{R}^+$

Thus the soln

$$y(t) = k_1 e^{w_0 it} + k_2 e^{-w_0 it} = C_1\cos(w_0 t) + C_2\sin(w_0 t), \quad C_1, C_2 \in \mathbb{R}$$

c. From b. we have that

$$y_h(t) = C_1\cos(w_0 t) + C_2\sin(w_0 t)$$

If $\alpha \neq w_0$ then for $g(t) = \sin(\alpha t)$ we have that

$$y(t) = y_h(t) + \underbrace{A\sin(\alpha t) + B\cos(\alpha t)}_{y_p(t)}, \quad A, B \in \mathbb{R}$$

If $\alpha = w_0$ then

$$y(t) = y_h(t) + At\sin(\alpha t) + Bt\cos(\alpha t)$$

(3) $my'' + \gamma y' + ky = g(t)$

Let $x_1 = y'$, $x_2 = y$ $\Rightarrow$ (3) $\Leftrightarrow$ $m x_1' + \gamma x_1 + k x_2 = g(t)$ $\Leftrightarrow$

$x_1' = y''$, $x_2' = y' = x_1$ $\Leftrightarrow$ $x_1' = \frac{g(t)}{m} - \frac{\gamma}{m} x_1 - \frac{k}{m} x_2$ (3')

Thus,

$$x_1' = -\frac{\gamma}{m} x_1 - \frac{k}{m} x_2 + \frac{g(t)}{m}$$

(Linear System of 1st order ODE's)

$$x_2' = x_1$$

e. See Attached

f. See Attached

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HOMEWORK 6

$\boxed{3}$ CONSIDER THE ORDINARY DIFFERENTIAL EQUATION:

$$y'' - y = 0 \qquad (4)$$

(a) SHOW THAT $y(x) = b_1 \sinh(x) + b_2 \cosh(x)$ IS A SOLUTION TO (4).

$y(x) = b_1 \sinh(x) + b_2 \cosh(x)$

$y'(x) = b_1 \cosh(x) + b_2 \sinh(x)$

$y''(x) = b_1 \sinh(x) + b_2 \cosh(x)$

$y'' - y = b_1 \sinh(x) + b_2 \cosh(x) - (b_1 \sinh(x) + b_2 \cosh(x)) = 0$

(b) SHOW THAT IF $c_1 = \frac{b_1 + b_2}{2}$ AND $c_2 = \frac{b_1 - b_2}{2}$ THEN
$y(x) = c_1 e^x + c_2 e^{-x} = b_1 \cosh(x) + b_2 \sinh(x)$

$y(x) = c_1 e^x + c_2 e^{-x}$

$= \left(\frac{b_1 + b_2}{2}\right) e^x + \left(\frac{b_1 - b_2}{2}\right) e^{-x}$

$= \frac{b_1(e^x + e^{-x})}{2} + \frac{b_2(e^x - e^{-x})}{2}$

$= b_1 \cosh(x) + b_2 \sinh(x)$

THE HYPERBOLIC SINE AND COSINE HAVE THE FOLLOWING TAYLOR SERIES REPRESENTATIONS:

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \qquad (5)$$

(c) ASSUME THAT $y(x) = \sum_{n=0}^{\infty} a_n x^n$ AND FIND THE GENERAL SOLUTION OF (4)

$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} a_n (n) x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$

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HOMEWORK 6

$y'' - y = 0$

$$\Rightarrow \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$K = n-2$ $\qquad$ $K = n$

$$\sum_{K=0}^{\infty} a_{K+2}(K+2)(K+1)x^K - \sum_{K=0}^{\infty} a_K x^K = 0$$

$$\sum_{K=0}^{\infty} \left[a_{K+2}(K+2)(K+1) - a_K \right] x^K = 0$$

SINCE x^K CAN'T BE 0, WE ASSUME

$a_{K+2}(K+2)(K+1) - a_K = 0$

$$\Rightarrow a_{K+2} = \frac{a_K}{(K+2)(K+1)} \quad K = 0, 1, 2, 3 \ldots$$

$K=0 \quad a_2 = \frac{a_0}{2 \cdot 1}$

$K=1 \quad a_3 = \frac{a_1}{3 \cdot 2 \cdot 1}$

$K=2 \quad a_4 = \frac{a_2}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}$

$K=3 \quad a_5 = \frac{a_3}{5 \cdot 4} = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$K=4 \quad a_6 = \frac{a_4}{6 \cdot 5} = \frac{a_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$K=5 \quad a_7 = \frac{a_5}{7 \cdot 6} = \frac{a_1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

FOR EVEN K: $\quad a_{2n} = \frac{a_0}{(2n)!} \quad n = 0, 1, 2, 3 \ldots$

FOR ODD K: $\quad a_{2n+1} = \frac{a_1}{(2n+1)!} \quad n = 0, 1, 2, 3 \ldots$

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{x^{(2n+1)}}{(2n+1)!}$$

$$\boxed{y(x) = a_0 \cosh(x) + a_1 \sinh(x)}$$