MACS 332 - August 3, 2006 NAME:
Quiz III - 30 minutes

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. Please enclose your final answers in boxes.

1. (10 Points) Conceptual question. Briefly describe the following mathematical concepts.

 a. The inner-product of $\mathbb{R}^n$.

 Let $\vec{x},\vec{y} \in \mathbb{R}^n$, the inner product of x, y is defined as $\vec{x}\cdot\vec{y} = \vec{x}^T\vec{y} = \sum_{i=1}^{n} x_i y_i \in \mathbb{R}$.
 The inner product measures the angle between vectors.
 If the vectors point in different "orthogonal" directions then $\vec{x}^T\vec{y} = 0$.

 b. The orthogonal complement of a vector space W.

 The orthogonal complement of a vector space W is a vector space of all vectors which are orthogonal to all vectors in W.

2. (10 Points) Let,

$$\mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

 Calculate the distance from $\mathbf{y}$ to the plane W=span$\{\mathbf{u}_1,\mathbf{u}_2\}$.

 Since $\vec{u}_1^T\vec{u}_2 = 0$,
 the projection of $\vec{y}$ onto the plane W is given by,

$$\hat{y} = \frac{\vec{y}^T\vec{u}_1}{\vec{u}_1^T\vec{u}_1}\vec{u}_1 + \frac{\vec{y}^T\vec{u}_2}{\vec{u}_2^T\vec{u}_2}\vec{u}_2 = \frac{-1+4}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1+4}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{5}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{3}{2} - \frac{5}{2} \\ \frac{3}{2} + \frac{5}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

 The distance is then given by.

$$\|\vec{y} - \hat{y}\| = \left\| \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\| = 3$$

3. (10 Points) Let $\mathbf{U}_{n\times n}, \mathbf{V}_{n\times n}$, be orthogonal matrices. Prove that $\mathbf{UV}$ is invertible and also an orthogonal matrix.

If U, V are orthogonal then their columns are mutually orthogonal and thus linearly independent. Since U, V are square with linearly independent columns, then they are invertible. Since the product of invertible matrices is invertible UV is invertible.

Lastly note that,

$$(UV)^T(UV) = (V^T U^T)(UV) = V^T(U^T U)V = V^T I V = V^T V = I.$$

implies that $(UV)^{-1} = (UV)^T$ and thus is orthogonal.

4. (10 Points) Given,

$$\mathbf{b}_1 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix}.$$

Let B= $\{\mathbf{b}_1, \mathbf{b}_2\}$ be a basis for the subspace W of $\mathbb{R}^3$. Determine an orthogonal basis for the subspace W.

Apply G.S. to $\{\vec{b}_1, \vec{b}_2\} = B$.

Let $\vec{V}_1 = \vec{b}_1$

then

$$\vec{V}_2 = \vec{b}_2 - \frac{\vec{b}_2^T \vec{V}_1}{\vec{V}_1^T \vec{V}_1}\vec{V}_1 = \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} - \frac{24-14}{16+4}\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} - \frac{10}{20}\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}$$

to give,

$$B_\perp = \left\{ \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix} \right\}$$

Which is an orthogonal Basis for W.