MATH348: SPRING 2012 - HOMEWORK 5

SIMPLE SOLUTIONS, CONSERVATION LAWS, HEAT EQUATIONS AND ALTERNATE COORDINATES

Even your emotions had an echo in so much space.

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ABSTRACT. Partial differential equations (PDE) relate the instantaneous rates of change of an unknown multivariate function and typically arise when modeling natural phenomenon that often, but not always, depend on space and time. The general theory of partial differential equations is a difficult topic and still an active area of research.[1] That said, we will investigate PDE through the eyes of separation of variables and its associated Fourier analysis. The moral will always be, Fourier series/transform provides a very general apparatus that allows one to represent a 'physically reasonable' function defined on a finite/infinite portion of space and, with this in mind, all we have to do is endow the Fourier representation with suitable dynamics consistent with the evolution defined by the PDE. Before we start this, it makes sense to consider where a PDE might come from.

The fundamental theorem of calculus coupled with the conservation of a quantity can be used to connect a flux to a density by the equation,

$$u_t - f + \nabla \cdot \vec{\phi} \tag{1}$$

where u is the density of the conserved quantity, $\vec{\phi}$ is its flux and f is a point-source function that captures any internal creation or destruction of the quantity. We close this equation by use of a constitutive relation relating, again, the flux to the density. This relation comes from empirical evidence. One such relation comes to us via the second law of thermodynamics, which says that the flux is proportional to the negative gradient of the density,

$$\vec{\phi} = -D\vec{\nabla}u \tag{2}$$

where D is known as the diffusivity.[2] Assuming a material with homogeneous properties now gives the diffusion equation,

$$u_t = D\triangle u + f, \tag{3}$$

which models the evolution of a density, in a homogeneous medium, that obeys the second law of thermodynamics. Now the question is, how do we solve this PDE and what does this tell us about the behavior of solutions? The following problems are meant to give us some ideas.

- P1. Before we get into it, we show some simple solutions to PDE. The point here is to remember our partial differentiation. Start with the given function, take the spatial and temporal derivatives the PDE calls for and show that the equality of the PDE is not broken by the function. The tricky one is 1.1, which asks to think really hard about the chain rule. We will discuss in class the physical meaning of this solution.
- P2. In this problem we go back to the conservation law but only consider the case in one-dimension of space. While, it is sensible to relate the flow to the density by the $2^{nd}-$law, it is not the only relation one can make. Here we assume instead $\phi \propto u$, which gives us a so-called transport equation, $u_t \propto u_x$. This equation predicts a different kind of solution. Namely, one that moves the data around without deformation. However, we can also think about the general equation $u_t = au_{xx} + bu_x + cu$, which is like a diffusion equation with lower order derivative terms. It turns out this equation can be mapped back onto the diffusion equation and so we choose to study just $u_t = Du_{xx}$.
- P3. Here we consider the problem outlined in class but now with different boundary conditions. From this we learn that the boundary conditions change the type of Fourier modes in the FS representation of the solution. Also, we can see that the diffusion equation tries to establish an equilibrium by averaging the initial data. This was not something we could see when the object was uninsulated. Lastly, we introduce a source term and outline how we can use what we learn in the homogeneous problem to solve the inhomogeneous problem.
- P4. It is asking too much to fix the background environment to a specific value. In this problem we consider a trick that allows us to offload a changing background to a source term. That is, if the boundary values are not 'standard' then we can make them 'standard' at the cost of an inhomogeneous term but we know how to solve the PDE with an inhomogeneous term from the last problem. Yay!
- P5. We now make the upgrade to multiple dimensions of space. Here I'm asking you to build off the example in class but the moral is, for each dimension of space you have another Fourier series *ad nauseum. I meant to assign this problem but didn't. You can do either this problem or problem 6. Doing both is worth extra credit.*
- P6. Lastly, we consider what happens to PDE when you convert to different coordinate systems. The answer is, nothing good. Here you will use the multivariate chain rule to represent the Laplacian in polar and spherical coordinates.

1. Some Solutions to common PDE

Show that the following functions are solutions to their corresponding PDE's.

1.1. **Right and Left Travelling Wave Solutions.** $u(x,t) = f(x-ct)+g(x+ct)$ for the 1-D wave equation.

1.2. **Decaying Fourier Mode.** $u(x,t) = e^{-4\omega^2 t}\sin(\omega x)$ where $c = 2$ for the 1-D heat equation.

1.3. **Radius Reciprocation.** $u(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ for the 3-D Laplace equation.

1.4. **Driving/Forcing Affects.** $u(x,y) = x^4 + y^4$ where $f(x,y) = 12(x^2+y^2)$ for the 2-D Poisson equation.

Note: The PDE in question are,

- Laplace's equation : $\triangle u = 0$
- Poisson's equation : $\triangle u = f(x,y,z)$
- Heat/Diffusion Equation : $u_t = c^2\triangle u$
- Wave Equation : $u_{tt} = c^2\triangle u$

and can be found on page 563 of Kryszig - 9^{th} Edition.

2. Conservation Laws in One-Dimension

Recall that the conservation law encountered during the derivation of the heat equation was given by,

$$\frac{\partial u}{\partial t} = -\kappa\nabla\cdot\boldsymbol{\phi} = -\kappa\,\mathrm{div}(\boldsymbol{\phi}), \tag{4}$$

which reduces to

$$\frac{\partial u}{\partial t} = -\kappa\frac{\partial\phi}{\partial x},\ \kappa\in\mathbb{R} \tag{5}$$

in one-dimension of space.[3] In general, if the function $u = u(x,t)$ represents the density of a physical quantity then the function $\phi = \phi(x,t)$ represents its flux. If we assume the ϕ is proportional to the negative gradient of u then, from (5), we get the one-dimensional heat/diffusion equation. [4]

2.1. **Transport Equation.** Assume that ϕ is proportional to u to derive, from (5), the convection/transport equation, $u_t + cu_x = 0\ \ c\in\mathbb{R}$.

2.2. **General Solution to the Transport Equation.** Show that $u(x,t) = f(x-ct)$ is a solution to this PDE.

2.3. **Diffusion-Transport Equation.** If both diffusion and convection are present in the physical system then the flux is given by, $\phi(x,t) = cu - du_x$, where $c,d\in\mathbb{R}^+$. Derive from, (5), the convection-diffusion equation $u_t + \alpha u_x - \beta u_{xx} = 0$ for some $\alpha,\beta\in\mathbb{R}$.

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[1]This is mostly due to the need to understand nonlinear equations, although the study of linear equations is not terribly easy. You might remember that nonlinear ordinary differential equation is difficult. So, you might suspect that the situation for PDE is at least as bad and probably worse. Well, you'd be right. In a sense PDE are always worse than ODE and this is because you can think of PDE as an infinite-dimensional generalization of ODE. That is, if the phase space of an ODE requires a finite number of degrees of freedom then a PDE, generally, requires an infinite number of degrees of freedom. This is bad.

[2]The diffusivity tells us how the physical properties of the material impede the flow. In mathematics we just write a number or function here but this hides a lot of science related to finding out the exact form of the diffusivity.

[3]When discussing heat transfer this is known as Fourier's Law of Cooling. In problems of steady-state linear diffusion this would be called Fick's First Law. In discussing electricity u could be charge density and q would be its flux.

[4]AKA Fick's Second Law associated with linear non-steady-state diffusion.

2.4. **Convection-Diffusion-Decay.** If there is also energy/particle loss proportional to the amount present then write the PDE where you have introduced the term λu to get the convection-diffusion-decay equation.[5]

2.5. **General Importance of Heat/Diffusion Problems.** Given that,

$$u_t = Du_{xx} - cu_x - \lambda u. \tag{6}$$

Show that by assuming, $u(x,t) = w(x,t)e^{\alpha x - \beta t}$, (6) can be transformed into a heat equation on the new variable w where $\alpha = c/(2D)$ and $\beta = \lambda + c^2/(4D)$.[6]

3. One Dimensional Heat Equation with Insulation and Source Term

Given,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} + F(x,t), \tag{7}$$

where $x \in (0,L)$ and $t \in (0,\infty)$, subject to

$$u_x(0,t) = 0, \;\; u_x(L,t) = 0, \tag{8}$$

and

$$u(x,0) = g(x). \tag{9}$$

3.1. **Homogeneous Case.** First let $F(x,t) = 0$ for all t and x. Solve the associated PDE via separation of variables and show that $\lim_{t\to\infty} u(x,t) = g_{\text{avg}} = \frac{1}{L}\int_0^L g(x)dx$.

3.2. **Cosine Half-Range Expansion.** Let $F(x,t) = e^{-t}\sin\left(\frac{2\pi}{L}x\right)$ be the heat generation function. Find the Fourier cosine half-range expansion of F.

3.3. **General Solution.** Using the previous result, solve for $G_n(t)$ for $n = 0,1,2,3,\dots$ assuming that $u(x,t) = G_0(t) + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L}x\right) G_n(t)$.

3.4. **Fourier Coefficients.** Assuming that $g(x) = \begin{cases} \frac{2k}{L}x, & 0 < x < \frac{L}{2}, \\ \frac{2k}{L}(L-x), & \frac{L}{2} < x < L \end{cases}$, solve for any unknown constants associated with the general solution.

4. Time Dependent Boundary Conditions

It makes sense to consider time-dependent interface conditions. That is, (7) and (9) subject to

$$u(0,t) = g(t), \;\; u(L,t) = h(t), \;\; t \in (0,\infty) \tag{10}$$

Show that this PDE transforms into:

$$\frac{\partial w}{\partial t} = c^2 \frac{\partial^2 w}{\partial x^2} - S_t(x,t) \quad , \tag{11}$$

$$x \in (0,L)\,, \qquad t \in (0,\infty)\,, \qquad c^2 = \frac{\kappa}{\rho\sigma}. \tag{12}$$

with boundary conditions and initial conditions,

$$w(0,t) = w(L,t) = 0, \tag{13}$$

$$w(x,0) = F(x), \tag{14}$$

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[5]The u_{xx} term models diffusion of energy/particles while u_x models convection, u models energy/particle loss/decay. The final term should not be surprising! Wasn't the appropriate model for radioactive/exponential decay $Y' = -\alpha^2 Y$?

[6]This shows that the general PDE (6), which models a flow that displays diffusive, transport and decay behaviors can be solved using heat equation techniques.

where $F(x) = f(x) - S(x, 0)$ and $S(x,t) = \dfrac{h(t)+g(t)}{L}x + g(t)$. [7]

5. Heat Equation on a spatially bounded domain of $\mathbb{R}^{2+1}$

Suppose that heat is allowed to flow in an $x, y-$plane, of finite area, $A = L_x L_y$, that has been insulated in the $z-$direction and its perimeter.

5.1. **Separation of Variables.** Find three ODEs consistent with the heat equation modeling the physical situation described above.

5.2. **Boundary Value Problems.** Write down the boundary conditions implied by the physical situation above and solve all ODEs, with their corresponding boundary conditions, given by the separation of variables above.

5.3. **Fourier Synthesis.** Apply superposition to the solutions of the ODE/BVPs from the previous step to find the general solution to the heat equation. From the general solution, show that the long-time behavior is to average the initial condition over the plane.

5. Coordinate Systems, Multivariate Chain Rule and the Laplacian

Recall that the Laplacian, $\triangle u = u_{xx} + u_{yy} + u_{zz}$, was a general term in the heat equation in $\mathbb{R}^{3+1}$. This is especially nice in Cartesian coordinates but if you change coordinates then the multivariate chain rule must be used to convert the associated derivatives. For example in polar coordinates $r = \sqrt{x^2+y^2}$ and $u_r(x,y) = u_r r_x + u_r r_y$. For this reason the Laplacian changes form in cylindrical and spherical coordinates.

5.1. **Laplacian in Cylindrical Coordinates.** Show that if $x = r\cos(\theta)$ and $y = r\sin(\theta)$ then $\triangle u = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz}$.

5.2. **Laplacian in Spherical Coordinates.** Show that if $x = \rho\cos(\theta)\sin(\phi)$, $y = \rho\sin(\theta)\sin(\phi)$ and $z = \rho\cos(\phi)$ then $\triangle u = u_{rr} + 2r^{-1}u_r + r^{-2}u_{\phi\phi} + r^{-2}\cot(\phi)u_\phi + r^{-2}\csc^2(\phi)u_{\theta\theta}$

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[7]A similar transformation can be found for the wave equation with inhomogeneous boundary conditions. The moral here is that time-dependent boundary conditions can be transformed into externally driven (AKA Forced or inhomogeneous) PDE with standard boundary conditions.