Note Title 2/4/2008

Spherical coords $\quad R(r)\,\Theta(\theta)\,\Phi(\varphi)$

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{k} + \underbrace{\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}_{-k} = 0$$

$$R(r)\ \begin{cases} r^{\ell} \\ r^{-(\ell+1)} \end{cases} \Rightarrow \text{gives } k$$

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\,\Theta\sin\theta = 0$$

Legendre poly's for ℓ integers

for $\ell = 0 \quad \Theta = \ln\left(\tan\left(\frac{\theta}{2}\right)\right) \longrightarrow \infty$ at $\theta = 0$

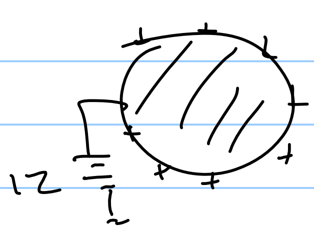
not useful

$$V(r,\theta) = A_\ell r^\ell P_\ell(\cos\theta) + B_\ell r^{-(\ell+1)} P_{-(\ell+1)}(\cos\theta)$$

$$P_\ell(\cos\theta) = P_{-(\ell+1)}(\cos\theta)$$

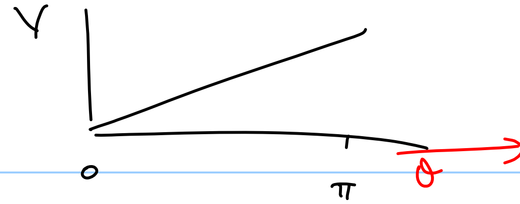
$$V(r,\theta) = \left[A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right] P_\ell(\cos\theta)$$

Ex!



$? \quad 12 = \left[A_\ell R + B R^{-(\ell+1)}\right] P_\ell(\cos\theta) \ ?$

Ex:



insulated stips at diff voltages

$V(r=R, \theta)$

General Soln

$$V(r,\theta) = \sum_{\ell} \left[A_\ell r^\ell P_\ell(\cos\theta) + B_\ell r^{-(\ell+1)} P_{-(\ell+1)}(\cos\theta) \right]$$

$$\int_{-1}^{1} P_\ell(x) P_m(x)\, dx = \int_0^\pi P_\ell(\cos\theta) P_m(\cos\theta) \sin\theta\, d\theta = \begin{cases} 0 & \ell \neq m \\ \frac{2}{2m+1} & \ell = m \end{cases}$$

$x = \cos\theta$

Ex: $V = V_0(\theta)$ at $\underline{r=R}$ find V everywhere

$$V = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-(\ell+1)} \right) P_\ell$$

for $r < R$ $\frac{1}{r^{\ell+1}} \xrightarrow{r \to 0} \infty \Rightarrow B_\ell = 0$

Inside $V_0(r=R,\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta)$

multiply both sides by $P_m(\cos\theta)\, d(\cos\theta)$ & integrate

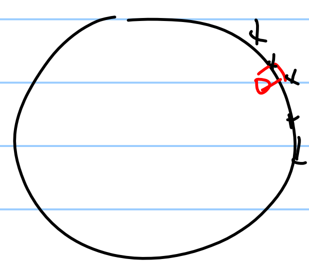
$$\int_{-1}^{1} V_0(\theta)\, P_m(\cos\theta)\, d(\cos\theta) = \sum_{l=0}^{\infty} A_l R^l \int_{-1}^{1} P_l(\cos\theta) P_m(\cos\theta)\, d(\cos\theta)$$

$$= \sum_{l=0}^{\infty} A_l R^l \frac{2}{2l+1}\delta_{lm} = A_m R^m \frac{2}{2m+1}$$

$x = \cos\theta$ $\quad$ $\theta : 0, \pi$ $\quad$ $\cos 0 = 1$ & $\cos\pi = -1$

$$A_l = \frac{2l+1}{2R^l}\int_{-1}^{1} V_0(\theta)\, P_l(\cos\theta)\, d(\cos\theta) \quad \text{for } r < R$$

$\vec{E} = -\vec{\nabla} V$ $\quad$ $\sigma(r=R, \theta)$ given by



$E_\perp = \frac{\sigma}{\epsilon_0}$

$\uparrow$

$-(\vec{\nabla} V)_\perp$

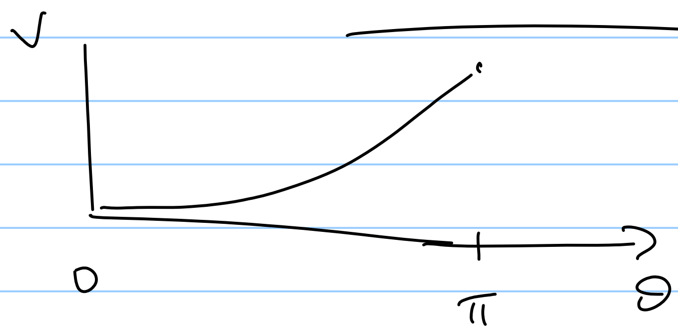
$r > R$ $\quad$ r can go to $\infty \Rightarrow A_l = 0$ so V does not go to ∞

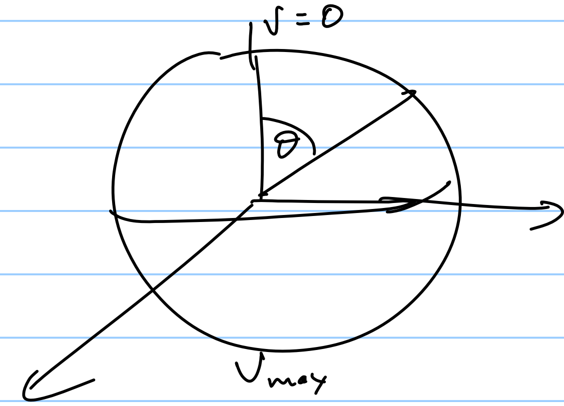
$$V = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right) P_l$$

$$V(r,\theta) = \sum B_l r^{-(l+1)} P_l(\cos\theta)$$

$$\underline{\underline{B_\ell}} = \frac{2\ell+1}{2} R^{(\ell+1)} \int_{-1} V_0(\theta) P_\ell(\cos\theta)\, d(\cos\theta)$$

Ex. say $\underline{V_0(\theta) = k \sin^2 \frac{\theta}{2}}$





$$V_0(\theta) = \frac{k}{2}(1 - \cos\theta) = k \sin^2\left(\frac{\theta}{2}\right)$$

$$= \frac{k}{2}\Big[\underbrace{P_0(\cos\theta)}_{\text{const}} - P_1(\underbrace{\cos\theta}_{x})\Big]$$

$$\int \frac{k}{2}\Big[P_0(\cos\theta) P_\ell(\cos\theta) - P_1(\cos\theta) P_\ell(\cos\theta)\Big] d(\cos\theta)$$

$$\int P_0 P_\ell \, d(\cos\theta) = \begin{matrix} 0 \\ \underline{\ell = 0} \end{matrix}$$

$$B_0 = \frac{2(0)+1}{2R^{(0)}} \int_{-1}^{1} \frac{k}{2} P_0 P_0 \, dx = \frac{1}{2}\frac{k}{2}\int_{-1}^{1} (1)^2 dx = \frac{k}{2}$$

$$B_1 = \frac{2(1)+1}{2R^1} \int_{-1}^{1} \frac{k}{2}(-P_1) P_1 \, dx = \frac{3}{2R}\frac{k}{2}\int -x^2 dx = +\frac{1}{2}\frac{k}{R}$$

$$V(r,\theta) = \sum B_\ell r^\ell P_\ell = B_0 P_0 + B_1 r P_1$$

$$= \frac{k}{2}(1) + \left(-\frac{1}{2}\frac{k}{R}\right) r\cos\theta$$