

Eigenvalues - Eigenvectors - Diagonalization - Spectral Decomposition - Applications

1. Given,

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

- (a) Determine the eigenvalues of \mathbf{A} .
- (b) Determine the eigenvectors of \mathbf{A} .

2. Given,

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}.$$

Determine the eigenvalues and eigenfunctions associated with the system of differential equations $\frac{d\mathbf{x}}{dt} = \mathbf{A} \cdot \mathbf{x}(t)$.

3. Given,

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

If \mathbf{A} is diagonalizable, then determine \mathbf{D} and \mathbf{P} associated with its decomposition $\mathbf{P}\mathbf{D}\mathbf{P}^{-1}$. Do not find \mathbf{P}^{-1} .

4. Square matrices having columns whose entries sum to 1 are often called stochastic matrices. Those with only non-negative entries, for some power, are called *regular* stochastic matrices. Given a random process, with an initial state \mathbf{x}_0 , the application of \mathbf{P} on \mathbf{x}_0 discretely steps the process forward in time. That is $\mathbf{x}_{n+1} = \mathbf{P}\mathbf{x}_n = \mathbf{P}^n\mathbf{x}_0$, $n = 1, 2, 3, \dots$. If a matrix is a *regular* stochastic matrix then there exists a steady-state vector \mathbf{q} such that $\mathbf{P}\mathbf{q} = \mathbf{q}$. This vector determines the long term probabilities associated with an arbitrary initial state \mathbf{x}_0 . The sequence of states, $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n+1}\}$, is called a *Markov Chain*. Given the regular stochastic matrix:

$$\mathbf{P} = \begin{bmatrix} .1 & .6 \\ .9 & .4 \end{bmatrix}.$$

- (a) Show that the steady-state vector of \mathbf{P} is $\mathbf{q} = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \end{bmatrix}^\top$.
- (b) Find the matrices \mathbf{D} and \mathbf{Q} such that $\mathbf{P} = \mathbf{Q}\mathbf{D}\mathbf{Q}^{-1}$. That is, diagonalize the matrix \mathbf{P} .
- (c) Show that $\lim_{n \rightarrow \infty} \mathbf{P}^n \mathbf{x}_0 = \mathbf{q}$ where $\mathbf{x}_0 = [x_1, x_2]^\top$ is an arbitrary vector in \mathbb{R}^2 such that $x_1 + x_2 = 1$.

5. Recall the Pauli Spin Matrix from homework 1,

$$\sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

- (a) Show that σ_y is self-adjoint.
- (b) Find the orthogonal diagonalization of σ_y .
- (c) Show that $\sigma_y = \lambda_1 \mathbf{x}_1 \mathbf{x}_1^\top + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^\top$, where \mathbf{x}_1 and \mathbf{x}_2 are the normalized eigenvectors from part (b).

5. Give $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$$\begin{aligned}
 \text{a) } \det(A - \lambda I) &= \det \begin{pmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{pmatrix} = \\
 &= (4-\lambda)\{(1-\lambda)(1-\lambda)\} + 1 \cdot \{-2(1-\lambda)\} = 0 \iff \\
 &\iff 4-\lambda\{(1-\lambda)^2\} = -2(1-\lambda), \quad \lambda=1 \Rightarrow 0=0 \\
 &\iff (4-\lambda)(1-\lambda) = -2 \quad \iff \\
 &\iff 4+\lambda^2-5\lambda+2=0 \quad \iff \\
 &\iff \lambda^2-5\lambda+6=0 \Rightarrow \lambda = 2, 3
 \end{aligned}$$

Thus, A has three eigenvalues,

$$\lambda_1 = 1$$

$$\lambda_2 = 2$$

$$\lambda_3 = 3$$

b)

Case $\lambda = 1$:

$$[A - \lambda I | 0] = \left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\Rightarrow -2x_1 = 0$$

$$3x_1 + x_3 \Rightarrow \vec{x} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$x_2 = \text{free}$

The basis for the eigenspace associated with $\lambda=1$ is,

$$B_{\lambda=1} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Case $\lambda=2$:

$$[A - \lambda I | 0] = \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{aligned} x_1 &= \frac{1}{2}x_3 \\ x_2 &= x_3 \\ x_3 &\text{ is free} \end{aligned} \quad \vec{x} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix} x_3$$

$$\text{The basis for this case is } B_{\lambda=2} = \left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix} \right\}$$

Case $\lambda=3$:

$$[A - \lambda I | 0] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow$$

$$\Rightarrow \begin{aligned} x_1 &= -x_3 \\ x_2 &= x_3 \\ x_3 &\text{ is free} \end{aligned} \Rightarrow \vec{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} x_3$$

\Rightarrow Basis for Eigenspace of A when $\lambda=3$ is

$$B_{\lambda=3} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

5. a. To find the eigenfunctions we find the eigenvalues and eigenvectors of A.

$$\det(A - \lambda I) = (3-\lambda)(1-\lambda) - (-2) = \lambda^2 - 4\lambda + 5$$

$$\lambda = \frac{-(-4) \pm \sqrt{16 - 4(1)(5)}}{2} = 2 \pm i$$

Case $\lambda = 2+i$:

$$[A - \lambda I | 0] = \left[\begin{array}{cc|c} 3-(2+i) & 1 & 0 \\ -2 & 1-(2+i) & 0 \end{array} \right] =$$

$$= \left[\begin{array}{cc|c} 1-i & 1 & 0 \\ -2 & -1-i & 0 \end{array} \right] \Rightarrow (1-i)x_1 + 1x_2 = 0$$

if $x \in \text{Null}(A - \lambda I)$
then this relationship
must hold.

$$\begin{aligned} \text{Let } x_1 = 1 &\Rightarrow \tilde{x} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix} \\ \Rightarrow x_2 = (1-i) & \end{aligned}$$

$$\text{Case } \lambda = 2-i \Rightarrow \tilde{x} = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

Thus the eigenfunctions of $\dot{\tilde{x}} = A\tilde{x}$ are given as,

$$\tilde{x}_1(t) = \begin{bmatrix} 1 \\ 1-i \end{bmatrix} e^{(2+i)t}, \quad \tilde{x}_2(t) = \begin{bmatrix} 1 \\ 1+i \end{bmatrix} e^{(2-i)t}$$

b. Using the formula pg 359 we have,

Real valued general soln

$$\tilde{x}(t) = C_1 \tilde{y}_1(t) + C_2 \tilde{y}_2(t)$$

$$= C_1 \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(t) \right\} e^{2t} +$$

$$+ C_2 \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(t) \right\} e^{2t}$$

3. Yes, A is diagonalizable. why? The following process will produce 4-linearly independent eigenvectors

1st: Since A is triangular we know the eigenvalues of A are

$$\lambda_1 = 4 \quad (\text{with algebraic multiplicity 2})$$

$$\lambda_2 = 2 \quad (\text{with algebraic multiplicity 2})$$

2nd:

Case $\lambda = 4$

$$[A - 4I | 0] = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \end{array} \right] \Rightarrow$$

$$\Rightarrow \begin{aligned} x_3 &= 0 \Rightarrow \bar{x} = \begin{bmatrix} 2x_4 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ x_1 &= 2x_4 \\ x_2, x_4 &\text{ free} \end{aligned}$$

Thus the Basis for the eigenspace is, $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = B_{\lambda=4}$

Case $\lambda = 2$

$$[A - 2I | 0] = \left[\begin{array}{cccc|c} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{aligned} x_1 &= 0 & \bar{x} &= \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ x_2 &= 0 & x_3 & \text{free} \\ x_3 & \text{free} & x_4 & \text{free} \end{aligned}$$

Basis for this Eigspace is, $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = B_{x=2}$

This implies that

$$P = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

2. a. Yes. The matrix is a stochastic matrix b/c its columns are probability vectors. It is Regular since P' has all nonnegative entries.

$$b. P\vec{q} = \vec{q} \Leftrightarrow (I - P)\vec{q} = 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} .9 & .6 & | & 0 \\ .9 & .6 & | & 0 \end{bmatrix} \sim \begin{bmatrix} .9 & .6 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\Rightarrow .9q_1 = .6q_2 \Rightarrow \vec{q} = \begin{bmatrix} 2/3q_2 \\ q_2 \end{bmatrix} = q_2 \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$$

q_2 is free

$$\text{Choose } q_2 \text{ to be } q_1 + \frac{2}{3}q_2 = 1 \Rightarrow q_2(\frac{2}{3}) = 1 \Leftrightarrow q_2 = \frac{3}{5}$$

$$q_2 = 3/5$$

\Rightarrow

$$\vec{q} = \begin{bmatrix} 6/15 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$$

Note

$$P\vec{q} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix} \checkmark$$

① Step 1 - Diagonalize P

$$\det \begin{pmatrix} 1-\lambda & .6 \\ .9 & .4-\lambda \end{pmatrix} = (.4-\lambda)(.1-\lambda) - .54 = \lambda^2 - .5\lambda - .54 + .4 = \lambda^2 - .5\lambda - .5 \Rightarrow \lambda_1 = 1$$

$$\lambda = \frac{-(-.5) \pm \sqrt{(-.5)^2 - 4(1)(-.5)}}{2(1)} = \frac{.5 \pm 1.5}{2} = 1, -.5$$

$$\lambda_1 = 1$$

$$\lambda_2 = -.5$$

$$\text{Case } \lambda=1, (P-I)x=0 \Rightarrow x = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$$

$$\text{Case } \lambda=-.5$$

$$(P + .5I)x = 0$$

\Rightarrow

$$\left[\begin{array}{cc|c} .6 & .6 & 0 \\ .9 & .9 & 0 \end{array} \right] \Rightarrow \bar{x} = x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus

$$P = \begin{bmatrix} 2/5 & 1 \\ 3/5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -.5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3/5 & 2/5 \end{bmatrix} \cdot \frac{1}{-.5}$$

and

$$P^k = \begin{bmatrix} 2/5 & 1 \\ 3/5 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & -.5^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3/5 & 2/5 \end{bmatrix}$$

Thus,

$$\lim_{k \rightarrow \infty} P^k = \begin{bmatrix} 2/5 & 1 \\ 3/5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3/5 & 2/5 \end{bmatrix} = \begin{bmatrix} .4 & .4 \\ .6 & .6 \end{bmatrix}$$

This implies that for $x_0 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x_1 + x_2 = 1$

$$\lim_{k \rightarrow \infty} x_k = \lim_{k \rightarrow \infty} B P^k B^{-1} x_0 = \begin{bmatrix} .4 & .4 \\ .6 & .6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .4(x_1 + x_2) \\ .6(x_1 + x_2) \end{bmatrix} = \begin{bmatrix} .4 \\ .6 \end{bmatrix}$$

4. Let $\hat{S}_Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

a. $\hat{S}_Y^H = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = \hat{S}_Y \Rightarrow \hat{S}_Y$ is self adjoint.

b. $\det(\hat{S}_Y^H - \lambda I) = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$$\frac{\lambda_1 = 1}{[\hat{S}_Y - I | 0]} = \left[\begin{array}{cc|c} 1 & -i & 0 \\ i & 1 & 0 \end{array} \right] \Rightarrow \begin{array}{l} x_1 - ix_2 = 0 \\ x_1 = ix_2 \end{array} \quad \bar{x} = \begin{bmatrix} -ix_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix} \Rightarrow$$

x_2 free

$$\Rightarrow \bar{x}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \quad \bar{v}_1 = \frac{1}{\|\bar{x}_1\|} \bar{x}_1 = \frac{1}{\sqrt{x_1^T \bar{x}_1}} \begin{bmatrix} -i \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ 1 \end{bmatrix} \quad (\text{normalized Eigenvector})$$

$\lambda_2 = -1$

$$[\hat{S}_Y + I | 0] = \left[\begin{array}{cc|c} 1 & -i & 0 \\ i & 1 & 0 \end{array} \right] \Rightarrow x_1 = ix_2 \quad \bar{x}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}, \bar{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1 \end{bmatrix}$$

x_2 free

Thus,

$$A = \begin{bmatrix} -i/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad P^H = \begin{bmatrix} -i/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$P = \begin{bmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad P^H = \begin{bmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} i_2 + i_2 & -i_2 + i_2 \\ i_2 - i_2 & i_2 + i_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$c. A = \lambda_1 \vec{v}_1 \vec{v}_1^H + \lambda_2 \vec{v}_2 \vec{v}_2^H =$$

$$= 1 \cdot \begin{bmatrix} -i/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix} \begin{bmatrix} +i/\sqrt{2} & i/\sqrt{2} \end{bmatrix} - 1 \cdot \begin{bmatrix} i/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix} \begin{bmatrix} -i/\sqrt{2} & i/\sqrt{2} \end{bmatrix} =$$

$$= \begin{bmatrix} +1/2 & -i/2 \\ +i/2 & i/2 \end{bmatrix} - \begin{bmatrix} +1/2 & i/2 \\ -i/2 & i/2 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$