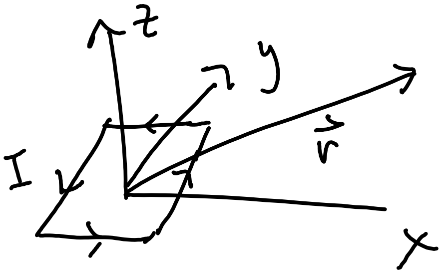
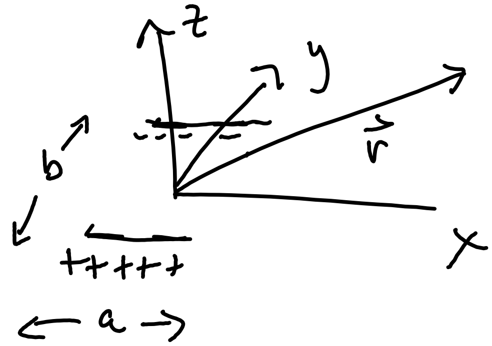
Homework 11 solutions

Homework problem 1.) Solve for the dipole vector potential in the x direction from a rectanglular wire carrying current I using a direct analogy with the electric dipole field from charge along one segment of this square.



analogy $\longrightarrow$



$$A_x = \frac{\mu_0}{4\pi}\int \frac{J_x\, d\tau'}{r}$$

$$= \frac{\mu_0}{4\pi}\int \frac{I\, dx'}{|\vec{r}-\vec{r}'|}$$

We want an approx for Ax far away.

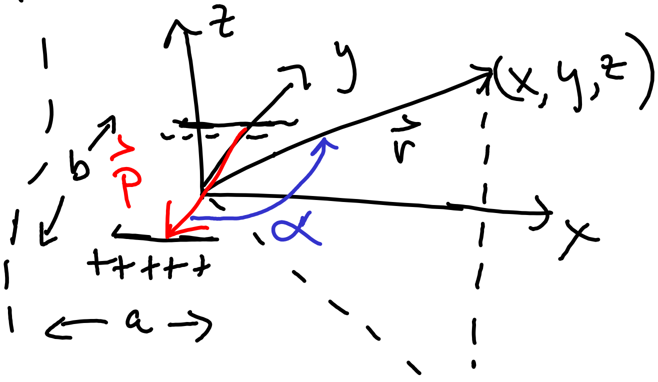
$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\, d\tau'}{r} = \frac{\lambda}{4\pi\epsilon_0}\int \frac{dx'}{|\vec{r}-\vec{r}'|}$$

We want an approx for V far away.

$$p = q\,b = \lambda a b$$

$$V_{dipole} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

$$\vec{p}\cdot\hat{r} = p\cos\alpha = p\left(-\frac{y}{r}\right)$$



The analogous relations are

$$\frac{\mu_0 I}{4\pi}\int \frac{dx'}{|\vec{r}-\vec{r}'|} \longleftrightarrow \frac{1}{4\pi}\frac{\lambda}{\epsilon_0}\int \frac{dx'}{|\vec{r}-\vec{r}'|}$$

$$V_{dipole} = \frac{1}{4\pi}\,\frac{\lambda a b\left(-y/r\right)}{\epsilon_0\, r^2}$$

$$A_{dipole} = \frac{1}{4\pi}\,\frac{I\mu_0 a b\left(-y/r\right)}{r^2}$$

$m = Iab$ is the magnetic dipole moment.

$$c^2 = \frac{1}{\mu_0\epsilon_0} \Rightarrow \mu_0 = \frac{1}{c^2\epsilon_0}$$

$$A_{dipole} = \frac{1}{4\pi} \frac{\mu_0 m(-y/r)}{r^2} = \frac{-1}{4\pi} \frac{m}{c^2 \epsilon_0} \frac{y}{r^3}$$

Homework problem 2: Derive an expression for the magnetic field from this dipole vector potential.

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

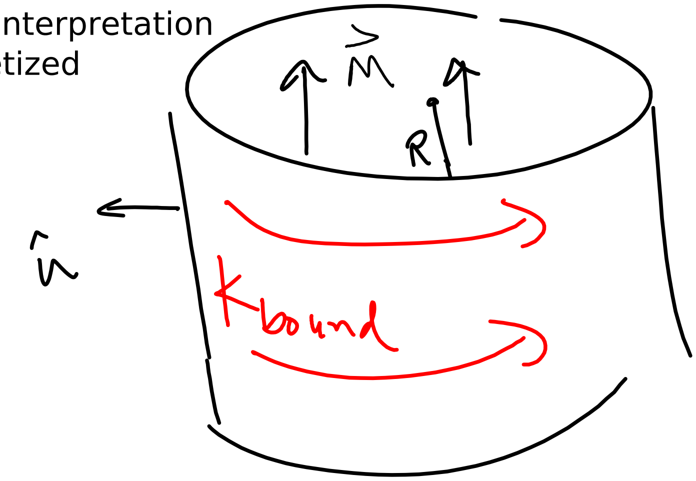
$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -\frac{\partial}{\partial z}\left(\frac{I a b x}{4\pi \epsilon_0 c^2 r^3} \right) = \frac{m}{4\pi\epsilon_0 c^2} \frac{3xz}{r^5}$$

$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = \frac{\partial}{\partial z}\left(-\frac{I a b y}{4\pi \epsilon_0 c^2 r^3} \right) = \frac{m}{4\pi\epsilon_0 c^2} \frac{3yz}{r^5}$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \frac{\partial}{\partial x}\left(\frac{m}{4\pi\epsilon_0 c^2} \frac{x}{r^3} \right) - \frac{\partial}{\partial y}\left(\frac{m}{4\pi\epsilon_0 c^2} \frac{y}{r^5} \right)$$

$$= \frac{m}{4\pi\epsilon_0 c^2} \left(\frac{1}{r^3} - \frac{3z^2}{r^5} \right)$$

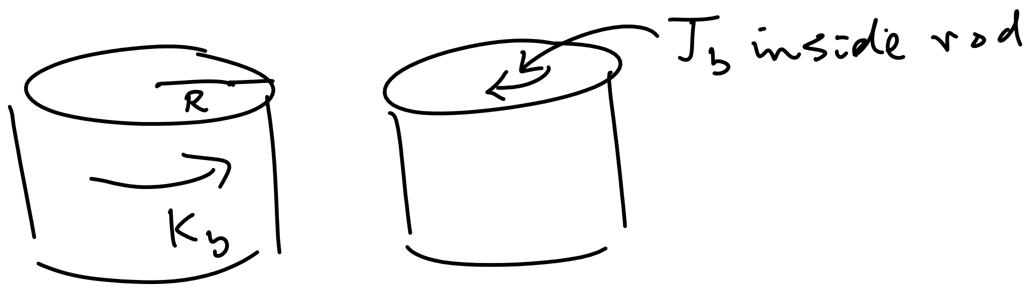
Here is the interpretation for a magnetized iron rod.

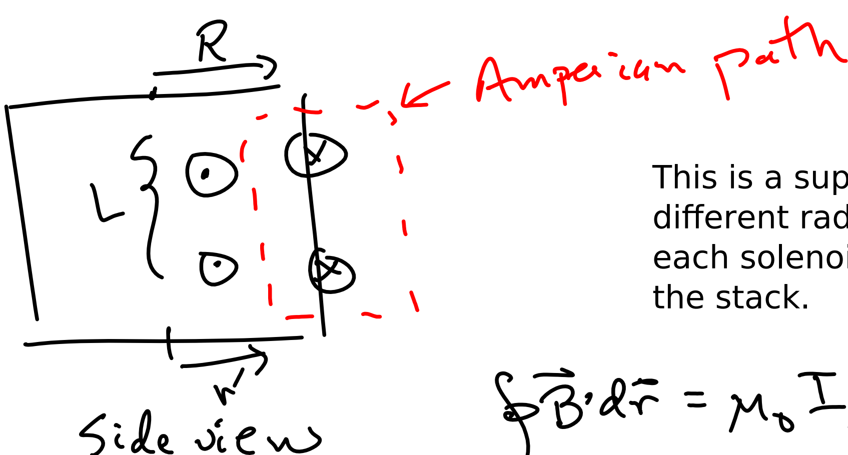


Homework problem 3.) If the rod above is very long and has $\vec{M} = \beta r \hat{z}$

(a) find both the surface and volume current densities.
(b) find the magnetic field.

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = -\beta \hat{\varphi} \qquad \vec{K}_b = \vec{M} \times \hat{n} = \beta R \hat{\varphi}$$





This is a superposition of solenoids of different radii stacked together. B outside each solenoid is zero so it is zero outside the stack.

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 I_{enclosed}$$

$$BL = \mu_0 \left\{ \int J_b \, da + K_b L \right\}$$

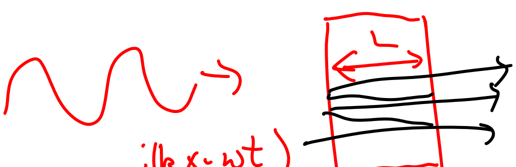
$$BL = \mu_0 \int_{r'}^{R} (-\beta L)\, dr + \mu_0 K_b L$$

$$= -\mu_0 \beta (R - r') L + \mu_0 K_b L$$

$$B = -\mu_0 \beta R L + \mu_0 \beta r' L + \mu_0 \beta R L = \mu_0 \beta r'$$

in the zhat direction

Homework problem 4.) "Perturbation" treatment of quantum well or light reflecting from the two boundaries.



$$E_{in} = E_0 e^{i(kx - \omega t)}$$

glass

$$\lambda \nu = \frac{c}{n} \qquad \omega = \frac{k_0}{n} c$$

$$k_0 = \frac{2\pi}{\lambda_{vacuum}}$$

r is the amplitude reflection coefficient and t is the amplitude transmission coefficient for each interface.

Find a series expansion for the electric field transmitted on the right interface (set x=0 at this interface). (b) sum this series to determine the transmitted electric field in terms of r and t.

The first term is for the EM wave going straight through. It has two transmissions through the two surfaces of the glass.

$$E_1 = E_0 t^2$$

The next wave is transmitted through the first surface, reflects from the second surface, reflects from the first surface, and is transmitted through the second surface.

$$E_2 = E_0 t r r t e^{i k_0 2 L n}$$

It also picks up a phase delay from having travelled the extra distance of twice L. This can be calculated as a retarded time or as kx.

$$t_{retarded} = t - \frac{2L}{c/n} \qquad \varphi = -\omega t = -\omega\left(t - \frac{2L}{c}n\right)$$

$$= -\omega t + \underbrace{\frac{\omega}{c}2Ln}_{\text{extra phase}}$$

$$\lambda_0 \nu = c \qquad \omega = k_0 c \qquad \frac{\omega}{c} = k_0 \qquad \Delta\varphi = k_0 2Ln$$

vacuum values

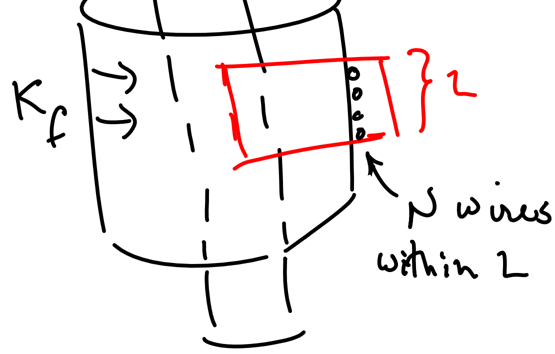
$$E_{tot} = E_1 + E_2 + \dots$$

$$= E_0 t^2 + E_0 t^2 r^2 e^{i\Delta\varphi} + E_0 t^2 r^4 e^{2i\Delta\varphi} + \dots$$

$$= \frac{E_0 t^2}{1 - r^2 e^{i\Delta\varphi}}$$

Homework problem 5.) A rod of copper (radius b) is placed in a solenoid n turns per length and current I0. What is B inside the copper? What is M and what are the bound currents?

$$B_{tot} = \mu_0 K_{free} + \mu_0 K_{bound} \quad \text{or}$$



$$\oint \vec{H} \cdot d\vec{r} = I_f$$

$$HL = N I_0$$

$$\vec{H} = \frac{N}{L} I_0 \hat{z} \qquad \vec{B} = \mu \vec{H} = \mu \frac{N}{L} I_0 \hat{z}$$

$$\vec{M} = \chi_m \vec{H} = \chi_m \frac{N}{L} I_0 \hat{z}$$

$$K_b = \underline{M} \times \hat{n} = \chi_m \frac{N}{L} I_0 \hat{\varphi}$$

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = 0 \quad \text{since } \vec{M} \text{ is constant}$$

Homework problem 6.) A thermonuclear weapon was detonated underground 30 years ago. The radioactive lifetime of the remants is thousands of years so the heat energy generated can be accurately modeled as being steady. It can be shown that the PDE for the steady state temperature distribution in the Earth is

$$k\nabla^2 T = g$$

where k is the thermal conductivity and g is the heat source energy per volume.

This is Poisson's equation. We can model the surface of the Earth above the detonation as not allowing any thermal energy to flow out of it.

(a) what parameters in electrostatics are the analogs of T and g?

(b) the flow of heat energy per time per area h is

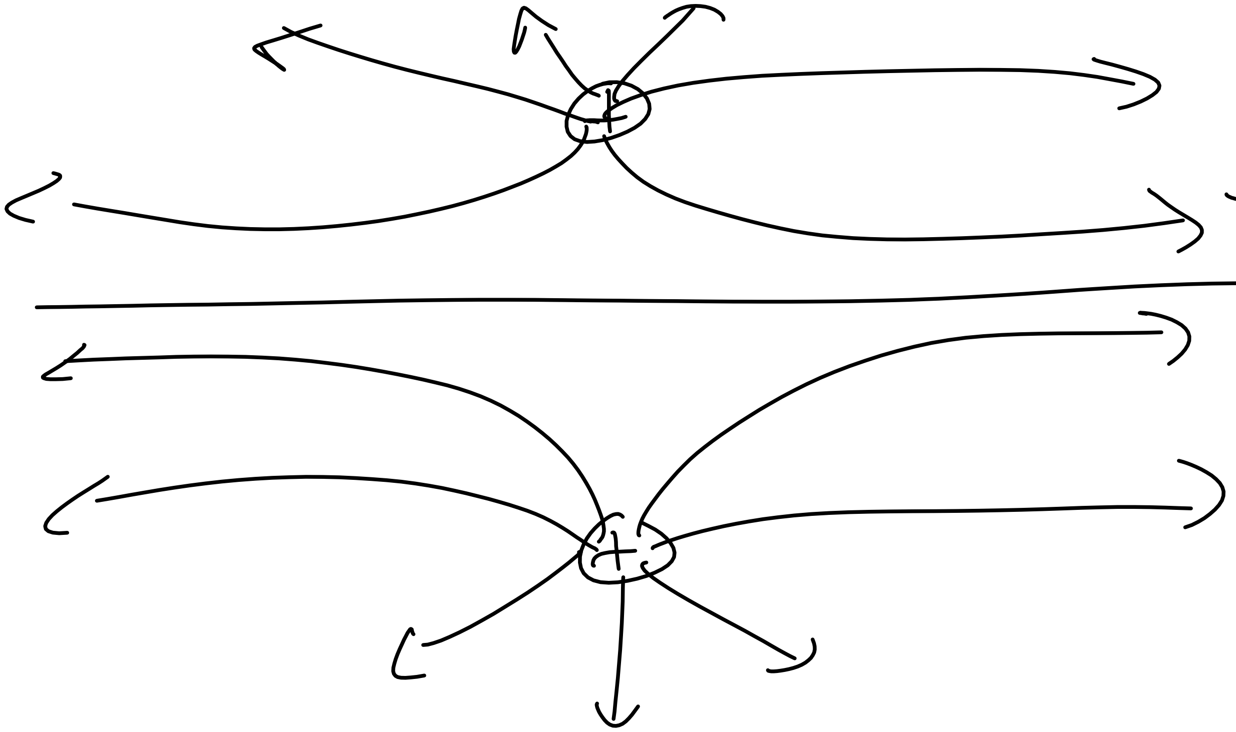
$$\vec{h} = -k\vec{\nabla} T$$

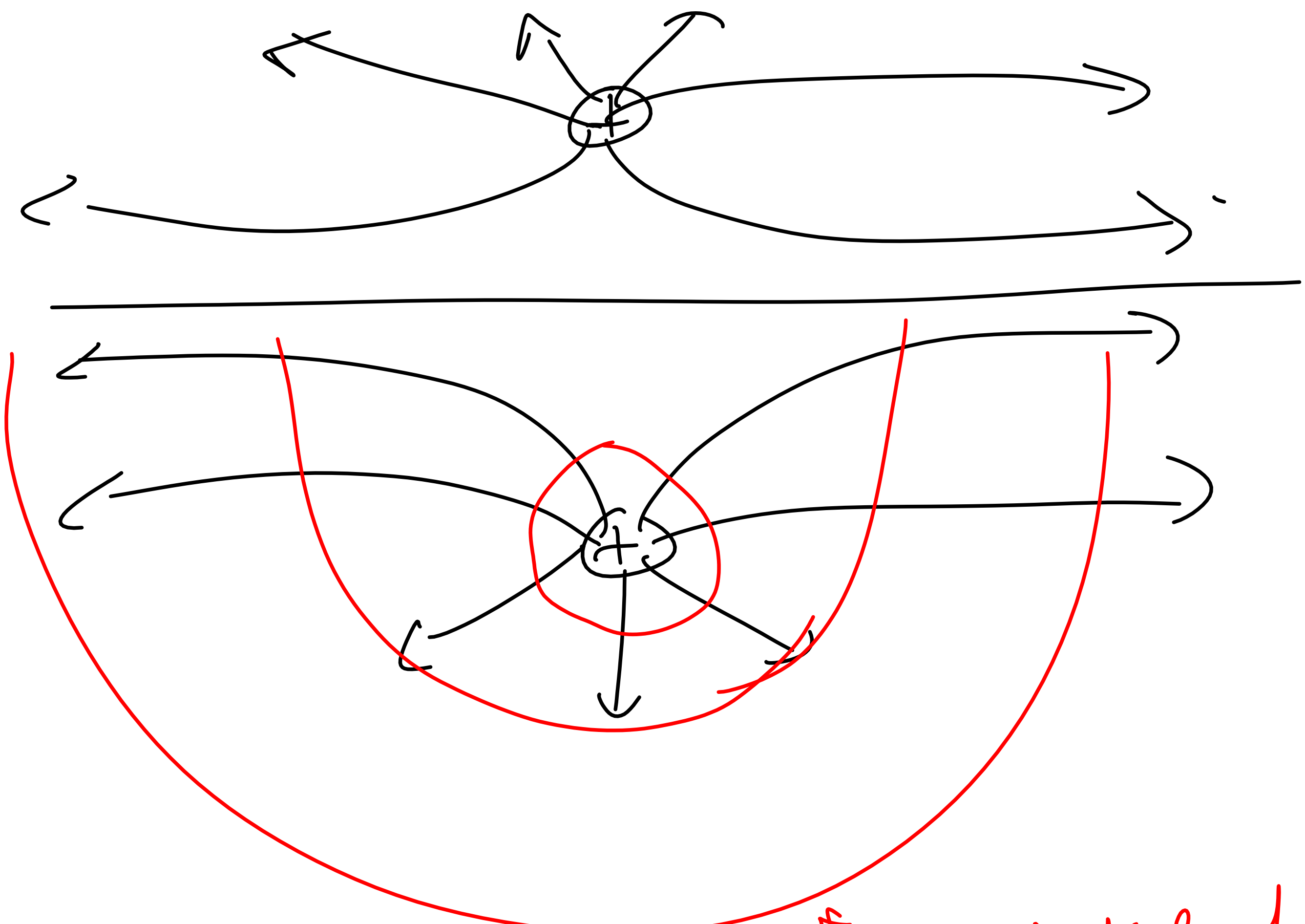
What is the analog of h in electrostatics?

(c) sketch the electrostatic analog (E lines and V contours) to this problem which satisfies the boundary conditions. Sketch also the thermal solution with lines of equal temperature and vector field lines of h.

The boundary condition is that the heat flow perpendicular to the surface is zero. This heat flow is analogous to the electric field. The source of the electric field is analogous to the source of heat (where the radioactive material is located).

congruous: How can I set up the electric fields from a charge distribution so they don't cross the surface?





equipotentials of
both voltage & temperature