difference between two measurements. (3) Most cheap sextants have some play in their gears. Consequently, one should always approach the contact point between the Moon and the Sun from the same direction and against the gear train; it is also a good idea not the change the sextant setting between two measurements since this will minimize residual play. (4) If the angular separation between the Sun and the Moon is large, one should avoid taking readings when one or the other is low in the sky: differential refraction will not be negligible.

This experiment always surprises the students who try it: after all, one does not usually associate a hand-held $50 gadget with fundamental astrometrical measurements. The experiment in fact shows the value of a "zero-measuring" device. The sextant owes its precision not merely to its 1 arcmin mechanical precision—this is actually quite easy to build in—but rather to the fact that the angle between two distinct objects is measured by superimposing their images using a single optical system: consequently, external perturbations affect both images equally and an accurate measurement can be obtained in spite of, for example, slightly shaky hands and without the need for massive supporting elements, leveling screws, etc. Moreover, by using differences between successive measurements, systematic instrumental errors are reduced or eliminated.

From the astronomical point of view, this experiment simulates the measurement of stellar parallaxes: the position of a star with respect to a background reference system of more distant "fixed" stars is measured—usually photographically—during the Earth's orbital motion around the Sun, but before obtaining a useful parallax it is necessary to allow for the intrinsic motion of the star itself. The same kinds of problems arise as in our sextant experiment; for example, one should not mix astrometrical results from different instruments.

And finally, three fundamental quantities are obtained rather easily during an afternoon's pleasant work in the sunshine!

On the approach to electro- and magneto-static equilibrium

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Many textbooks claim that the relaxation time for the approach to electrostatic equilibrium is ϵ_0/σ. We show that, for a good conductor, this claim is false. For such a conductor, the approach to the electro- and magneto-static equilibrium hinges on the damping of the induced dynamic electric and magnetic fields. The relaxation time depends on the conductivity, the geometry of the conductor, and the details of the initial charge distribution.

I. INTRODUCTION

As is well known, in a homogeneous conductor in electrostatic equilibrium, the electric field is zero and all the free charges reside on the surface. A question frequently asked in electricity textbooks is: how long does the conductor take to achieve this equilibrium configuration if the charges are initially distributed all over the volume? The authors of many introductory and even some advanced textbooks commit a glaring error in their attempts at answering this question.[1] They commonly present the following "derivation" of the characteristic relaxation time for the electrostatic field, taking as starting point the continuity equation

$$\nabla\cdot\mathbf{j}+\frac{\partial\rho}{\partial t}=0, \tag{1}$$

Ohm's law

$$\mathbf{j}=\sigma\mathbf{E}, \tag{2}$$

and Gauss' law

$$\nabla\cdot\mathbf{E}=\rho/\epsilon_0. \tag{3}$$

They substitute Eq. (2) into Eq. (1) and then use Eq. (3) to obtain a differential equation for the charge density

$$\frac{\sigma}{\epsilon_0}\rho+\frac{\partial\rho}{\partial t}=0, \tag{4}$$

which has the solution

$$\rho=\rho_0 e^{-(\sigma/\epsilon_0)t}. \tag{5}$$

From this, they jump to the conclusion that the relaxation time for achieving electrostatic equilibrium is

$$\tau_R=\epsilon_0/\sigma. \tag{6}$$

For a good conductor, e.g., for copper with a conductivity $\sigma=1/(1.7\times10^{-8}\,\Omega\cdot\mathrm{m})$, this expression yields an extremely short relaxation time, $\tau_R=1.5\times10^{-19}$ s.

But a bit of thought immediately convinces us that this expression for the relaxation time cannot be right—it has a nonsensical dependence on the relevant physical parameters. For a very good conductor, the relaxation time ought to be very long because, in the absence of dissipative forces, the free charges will surge back and forth on the conductor instead of settling quickly into their equilibrium positions. Thus, we expect that the relaxation time ought to increase with the conductivity. We also expect, on the basis of mechanical analogies, that the relaxation time ought to increase with the size of the conductor. Equation (6) fails to

meet these expectations. Furthermore, the numerical value of 10^{-19} s for the relaxation time is much too short and leads to an inconsistency with the physical assumptions underlying Ohm's law. On a time scale of 10^{-19} s, Ohm's law is *not valid*. This law hinges on setting up an average balance between the momentum gained by each free electron from the electric field and the momentum lost in collisions. Since the collision time τ for an electron in copper is about 2×10^{-14} s, it is obvious that a time *at least* that long is required to achieve steady-state conditions in the momentum balance.

In an attempt at improving on the naive calculation of the relaxation time, Saslow and Wilkinson[2] proposed the replacement of the simple proportionality between the instantaneous current density $\mathbf{j}(t)$ and the instantaneous electric field $\mathbf{E}(t)$ by a proportionality between the Fourier components $\mathbf{j}(\omega)$ and $\mathbf{E}(\omega)$, with a frequency-dependent conductivity

$$\mathbf{j}(\omega)=\sigma(\omega)\mathbf{E}(\omega). \tag{7}$$

Combining this with Eq. (1) and (3), one finds that the motion of the free-electron gas in the conductor gives rise to damped plasma oscillations with a relaxation time which is roughly equal to twice the collision time, i.e., about $2\times2\times10^{-14}$ s for copper. Within this time, the free charges will be expelled from the volume of the conductor.

However, the expulsion time of the free charges is *not* the same thing as the overall relaxation time of the conductor—there are other, longer-lasting, transient phenomena in the conductor. The basic error in both the naive calculation and the Saslow–Wilkinson calculation lies in neglecting the dynamics of the electric and magnetic fields. The transient current and displacement current in the conductor generate a time-dependent magnetic field, which induces an electric field, etc. Thus, in the analysis of the approach to equilibrium we must give due consideration to *all* of Maxwell's equations. As we will see, the expulsion of the fields and currents from the volume of the conductor is a diffusion process with a relaxation time much longer than that for the expulsion of charge. But this is not yet the end of the story. After the fields and currents have been expelled from the volume of the conductor, the charges must still seek their equilibrium position on the surface—the charges will oscillate, surging back and forth along the surface. These oscillating surface currents and the corresponding electric and magnetic wave fields that they set up in the space surrounding the conductor will be damped gradually by Ohmic and radiative losses.

We can therefore think of the approach to equilibrium as a three-step process: first, the free charges are expelled from the volume; second, the currents and the dynamic electric and magnetic fields are expelled from the volume; and third, the surface currents and the wave fields are damped. Of course, these three relaxation processes overlap to some extent; but for a rough estimate of the overall relaxation time we may take the sum of the individual relaxation times. Since the first step is very quick, the time scale for the approach to equilibrium is dominated by steps two and three.

II. EXPULSION OF TRANSIENT FIELDS

The problem of the expulsion of transient fields from the volume of a conductor has been explored in considerable detail because of its important practical applications, such as flux compression by conducting liners (see Ref. 3 for a list of further practical applications and further references). The following straightforward estimate of the relaxation time is due to Wilhelm.[3] The starting point consists of Maxwell's equations

$$\nabla\cdot\mathbf{E}=0,\quad \nabla\cdot\mathbf{B}=0, \tag{8}$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}, \tag{9}$$

and

$$\frac{1}{\mu_0}\nabla\times\mathbf{B}=\epsilon_0\frac{\partial\mathbf{E}}{\partial t}+\mathbf{j}. \tag{10}$$

If we take the curl of Eq. (10) and eliminate the current and the electric field by means of Eqs. (2) and (9), we readily obtain

$$\frac{\partial^2\mathbf{B}}{\partial t^2}+\frac{\sigma}{\epsilon_0}\frac{\partial\mathbf{B}}{\partial t}=\frac{1}{\mu_0\epsilon_0}\nabla^2\mathbf{B}. \tag{11}$$

In a good conductor, σ is very large; hence the first term on the left side of Eq. (11) can be neglected compared to the second. This leads to a parabolic differential equation,

$$\sigma\frac{\partial\mathbf{B}}{\partial t}=\frac{1}{\mu_0}\nabla^2\mathbf{B}. \tag{12}$$

A similar argument leads to a similar equation for the electric field (if $\rho=0$),

$$\sigma\frac{\partial\mathbf{E}}{\partial t}=\frac{1}{\mu_0}\nabla^2\mathbf{E}. \tag{13}$$

These are diffusion equations, analogous to the equation for heat conduction. They indicate that the transient magnetic and electric fields will "flow" out of the volume of the conductor in some characteristic diffusion time τ_D. With the crude estimates of $\partial B/\partial t\sim B/\tau_D$ and $\nabla^2B\sim B/l^2$ (where l is a characteristic length of the conductor) we find

$$\tau_D\sim\mu_0\sigma l^2. \tag{14}$$

This expression for the relaxation time has a reasonable dependence on the relevant parameters—it increases with the conductivity and it increases with the size of the conductor. It also yields reasonable numbers. For a copper sheet of thickness 1 mm, we obtain $\tau_D\sim10^{-4}$ s.

Incidentally, note that in a poor conductor, for which the calculation of Sec. I is valid, the electric field has the time dependence $e^{-(\sigma/\epsilon_0)t}$. From this it follows that the sum of the real current plus the displacement current is zero,

$$\epsilon_0\frac{\partial\mathbf{E}}{\partial t}+\mathbf{j}=\epsilon_0\frac{\partial\mathbf{E}}{\partial t}+\sigma\mathbf{E}=0. \tag{15}$$

Hence in a poor conductor, the approach to equilibrium does not involve any magnetic fields nor any induced electric fields. The existence of such dynamic fields in a good conductor hinges on the initial failure of Ohm's law.

III. DAMPING OF SURFACE CURRENTS AND WAVE FIELDS

Next, we must deal with the third step in the approach to equilibrium. The calculation of the relaxation time for the surging surface currents is a rather difficult problem requiring careful consideration of the electric and magnetic wave fields surrounding the conductor. One method of calculation relies on the normal modes of these electric and magnetic wave fields. Assuming that the normal modes are

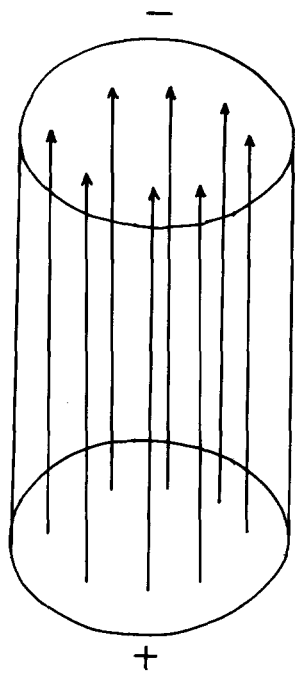

Fig. 1. Initial electric field in a cylindrical evacuated can.

known, we can expand the initial electric and magnetic fields associated with the initial surface charge and current configuration in terms of the normal-mode fields. The frequencies and the damping times of the normal modes then determine the time dependence of the field. Usually, the lowest-frequency normal mode has the longest damping time. Thus, we expect that the relaxation time for a typical initial configuration is equal to the damping time of the lowest-frequency mode. The period of this mode is of the order of l'/c (where l' is another characteristic length of the conductor) and the damping time is the period multiplied by the "Q" of the mode. This gives us a rough estimate for the relaxation time,

$$\tau_S \sim Ql'/c. \tag{16}$$

The "Q" of the mode depends on the geometry of the conductor and on the conductivity; it increases with conductivity if the Ohmic losses dominate over the radiative losses.

We can make this more precise by concentrating on a specific example, say, a conductor in the shape of a closed, cylindrical, evacuated can (see Fig. 1). If we initially have a uniform positive charge distribution on the interior of the lower lid and an equal negative charge distribution on the interior of the upper lid, the charge will surge up and down flowing along the lateral walls. These oscillations will gradually damp out, and the conductor will ultimately settle into electrostatic equilibrium. The given initial conditions correspond to a TM mode with frequency[4]

$$\omega = 2.4\, c/R \tag{17}$$

and a "Q"

$$Q = \frac{1}{2\pi}\frac{h}{1+h/R}\left(\frac{\omega\sigma\mu_0}{2}\right)^{1/2}. \tag{18}$$

The relaxation time is

$$\tau_S = Q(2\pi/\omega) = Q(2\pi R/2.4c) \tag{19}$$

or

$$\tau_S = [h/(1+h/R)](\sigma\mu_0 R/4.8c)^{1/2}. \tag{20}$$

This expression again has a reasonable dependence on the relevant parameters and it yields reasonable numbers. For an empty "beer can" with $h = 12$ cm, $R = 3.3$ cm made of copper, we obtain $\tau_S = 1.1\times10^{-6}$ s.

Note that for a very long, thin cylinder ($h \gg R$), the relaxation time is

$$\tau_S \cong R^{3/2}(\sigma\mu_0/4.8c)^{1/2}, \tag{21}$$

i.e., it is independent of h. Equation (21) has an interesting consequence: if h is sufficiently large, say $h = 1$ km and $R = 3.3$ cm, the relaxation time (21) will be *shorter* than the light-travel time h/c. This is not in conflict with causality because no signal needs to travel from one end of the conductor to the other—the signal is already present at the initial instant, in the electric fields of the initial charge configuration.

To estimate the overall relaxation time of the conductor we must select the larger of the two relaxation times τ_D and τ_S. For instance, if the above "beer" can is made of copper sheet 1 mm thick, then $\tau_D \gg \tau_S$; hence the expulsion of the fields from the volume of copper sets the time scale for the approach to equilibrium, while the damping of the wave fields can be neglected. But if the copper sheet had a thickness of, say, 10^{-2} mm, then $\tau_D \sim \tau_S$ and both steps would play a significant role. In any case, the approach to electrostatic and magnetostatic equilibrium takes much longer than many textbooks would have us believe.

[1]The error is to be found in D. R. Corson and P. Lorrain, *Introduction to Electromagnetic Fields and Waves* (Freeman, San Francisco, 1962), p. 191; J. R. Reitz and F. J. Milford, *Foundations of Electromagnetic Theory* (Addison-Wesley, Reading, MA, 1960), p. 139; Paul A. Tipler, *Physics* (Worth, New York, 1976), p. 801; R. K. Wangsness, *Electromagnetic Fields* (Wiley, New York, 1979), p. 241; J. B. Marion, *Classical Electromagnetic Radiation* (Academic, New York, 1965), pp. 105, 143; and W. K. Panofsky and M. Philipps, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, MA, 1962), p. 123. Much the same error, under the guise of a time dependence of the longitudinal electric field, is to be found in J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), p. 223. The oldest text in which I have seen this error is J. Jeans, *The Mathematical Theory of Electricity and Magnetism* (Cambridge U. P., London, 1908, reprinted 1960), p. 359; in fairness, I have to admit that Jeans appears to be mainly, though not exclusively, concerned with dielectrics (poor conductors), and for these Eq. (6) would actually be correct. Later editions of Corson and Lorrain (1970) and of Reitz and Milford (1979) confess that the naive derivation of the relaxation time [see Eqs. (1)–(6)] is not applicable to good conductors and they claim that the relaxation time in metallic conductors is roughly equal to the collision time, $\sim 10^{-14}$ s. This is a step in the right direction, but the claim that the relaxation time is anywhere near the collision time is completely false.

[2]W. M. Saslow and G. Wilkinson, Am. J. Phys. **30**, 1244 (1971).

[3]H. E. Wilhelm, J. Math. Phys. **23**, 10 (1982).

[4]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), pp. 254–258.