Note Title 10/1/2007

$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$A\vec{x} = \lambda\vec{x}$ $\quad A \in R^{2\times2}$

$\vec{x} \in R^2$

λ scalar

$\Rightarrow (A - \lambda I)\vec{x} = 0$

in order that $\vec{x}$ be in the null space of A, we must have

$\mathrm{Det}(A - \lambda I) = 0$

in this example

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix}$$

$\mathrm{Det}(A - \lambda I) = (1-\lambda)(1+\lambda) + 1 = 0$

$\Rightarrow -\lambda^2 + 1 + 1 = 0$

$\Rightarrow \lambda^2 = 2 \quad \sqrt{\lambda} = \pm\sqrt{2}$

Now for the $\vec{x}$ vector

$A\vec{x} = \lambda x$

Case 1 $\quad \lambda = \sqrt{2}$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \sqrt{2}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \quad x + y = \sqrt{2}\, x$$
$$x - y = \sqrt{2}\, y$$

$$x = (1+\sqrt{2})\, y$$

plug in for x

$$(1+\sqrt{2})\cancel{y} + \cancel{y} = \sqrt{2}(1+\sqrt{2})\cancel{y}$$

$$1+\sqrt{2} + 1 = \sqrt{2} + 2$$

identity does not constrain y.

so pick y and then
$$x = (1+\sqrt{2})\, y$$

$$\vec{x} = \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}$$

Case 2 $\quad \lambda = -\sqrt{2}$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\sqrt{2}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \quad x + y = -\sqrt{2}\, x$$
$$x - y = -\sqrt{2}\, y$$

$x = (1-\sqrt{2})\, y$, as before

$$\vec{x} = \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}$$

matrix of ε-vectors

$$\begin{pmatrix} 1+\sqrt{2} & 1-\sqrt{2} \\ 1 & 1 \end{pmatrix}$$

Diagonal matrix of ε-values

$$\begin{pmatrix} \sqrt{2} & \\ & -\sqrt{2} \end{pmatrix}$$

Ex. $\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$ symmetric

char. polynomial

$(5-\lambda)(2-\lambda) - (-2)(-2) = 0$

$\lambda^2 - 7\lambda + 10 - 4 = 0$

$\lambda^2 - 7\lambda + 6 = 0$

$\lambda_\pm = \frac{7}{2} \pm \frac{1}{2}\sqrt{49 - 24}$

$= \frac{7}{2} \pm \frac{5}{2} = 6, 1$

Now compute the E-vectors

we know that

$A\vec{x} = \lambda_\pm \vec{x}$ so,

solve this for each λ

$$\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 6 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$5x - 2y = 6x$$

$$\Rightarrow -2y = x$$

$$\text{const} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

similarly

$$5x - 2y = x$$

$$\Rightarrow -2y = -4x$$

$$y = 2x$$

$$c \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

E-vectors are only det. up to a constant

$$A\vec{x} = \lambda \vec{x}$$

$$\Rightarrow \quad A(\alpha \vec{x}) = \lambda(\alpha \vec{x})$$

So our E-vectors are

$$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$ OK but INCONV

Take advantage of the nonuniqueness to normalize

$$\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

Length of each column is 1

TRY this

$$\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}^T \begin{pmatrix} 6 & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} = ?$$

Let $\vec{x}_1$ be the ε-vector assoc. with λ_1

$\{\vec{x}_1, \lambda_1\}$, $\{\vec{x}_2, \lambda_2\}$

ε-value, ε-vector pairs

$$A\vec{x}_1 = \lambda_1 \vec{x}_1$$

$$A\vec{x}_2 = \lambda_2 \vec{x}_2$$

become columns

$$A\begin{pmatrix} | & | \\ \vec{x}_1 & \vec{x}_2 \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ \vec{x}_1 & x_2 \\ | & | \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Call this

$$AC = \Lambda C$$

$\uparrow$ diagonal matrix

then if C^{-1} exists we have

$$\boxed{C^{-1} A C = \Lambda}$$

matrix diagonal-ization

C gives a change in coordinates in which A is diagonal

matrices [illegible]
Linear transformations

in general

$$C^{-1} M C = D$$

is a similarity transf.

See p. 151 Boas

Symmetric matrices

$$A = A^T$$

Hermitian matrices

$$H = \overline{H^T} \equiv H^\dagger$$

Symm. matrices can be diagonalized by orthogonal matrices w/ real ε-values

$$Q^T Q = I$$

Hermitian matrices can diagonalized by unitary matrices w/ real ε-values

$$U^\dagger U = I$$

For an $n \times n$ matrix, the charastic polynomial is n^{th} order.

Finding the root is hard then you

still must solve n Linear systems for the ξ-vectors!

Suppose $Q^T Q = I$

$$\|Q\vec{x}\|^2 = (Qx, Qx)$$

$$= (x, Q^T Q x)$$

$$= (x, x) = \|\vec{x}\|$$

mathematic exapes