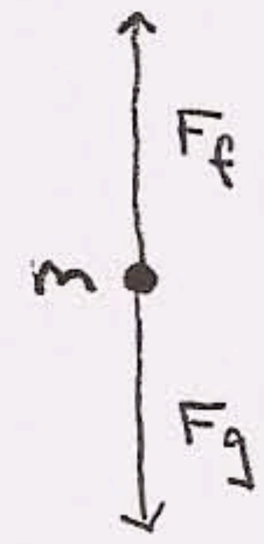
MATH 225 - May 15, 2008 NAME: Key

Quiz I - 20 minutes

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

Consider the free body diagram for an object in free-fall:



$F_g \equiv$ Force do to gravity

$F_f \equiv$ Force do to friction.

1. Noting that $F_{net} = ma$ and assuming that the force due to friction is proportional to the velocity of the particle, derive a first-order ODE, which models the velocity, v of the mass as a function of t.

$$F_{net} = ma = m\frac{dv}{dt} = F_g - F_f = mg - bv$$

2. Using separation of variables solve the previous ODE and assuming that the initial velocity of the mass m is given by $v(0) = \frac{gm}{b}$, where b is the coefficient of kinetic friction, and solve for any unknown constants.

$$\int \frac{m}{mg - bv}\, dv = \int dt = t + C$$

$$-\frac{m}{b} \ln|mg - bv| = t + C \Rightarrow mg - bv = e^{-\frac{b}{m}t + C} \Rightarrow V(t) = \frac{mg}{b} - \frac{C}{b} e^{-\frac{b}{m}t} : \text{general Soln}$$

$$V(0) = \frac{gm}{b} = \frac{mg}{b} - \frac{C}{b} e^{0} \Rightarrow C = 0 \quad V(t) = \frac{mg}{b} \text{ soln to IVP}$$

3. By direct substitution show that your solution satisfies your ODE.

$$V'(t) = 0 \Rightarrow mg - bV(t) = mg - b\frac{mg}{b} = 0 \checkmark$$

Yes it is a soln

4. Now assume that the force of friction is proportional to the square of velocity. Write down and ODE, which models the velocity, v of the mass, m, as a function of time, t.

$$m\frac{dv}{dt} = mg - bv^2$$

5. Let $m = b = 2$ and $g = 1$ then using separation of variables find the explicit solution to the ODE from question 4.

$$\text{If } m = b = 2, g = 1 \Rightarrow \frac{dv}{dt} = 1 - v^2 \Leftrightarrow \int \frac{dv}{v^2 - 1} = \int -dt = -t + C$$

where

$$\frac{1}{v^2 - 1} = \frac{A}{v+1} + \frac{B}{v-1} = \frac{A(v-1) + B(v+1)}{v^2 - 1} \Rightarrow \begin{matrix} A = -\frac{1}{2} \\ B = \frac{1}{2} \end{matrix} \quad \text{thus } \int \frac{dv}{v^2 - 1} = \int \left(\frac{-1}{2}\frac{1}{v+1} + \frac{1}{2}\frac{1}{v-1} \right) dv =$$

$$= \ln\left| \frac{(v-1)^{1/2}}{(v+1)^{1/2}} \right|$$

Hence

$$\frac{v-1}{v+1} = Ce^{-2t} \Rightarrow v(t) = \frac{Ce^{-2t} + 1}{1 - Ce^{-2t}}$$

$C \in \mathbb{R}$

Sanity Check

$\lim_{t \to \infty} v(t) = 1$ which is the Equilibrium Soln.