

Note Title

2/12/2007

Laplace's eqn in spherical coord)

$$\nabla^2 V = 0 \quad \text{multiply by } r^2 \sin^2 \theta$$

Let $V = R(r) \Theta(\theta) \Phi(\phi)$ divide by V

$$\frac{\sin^2 \theta}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = 0$$

$$f(r, \theta) - m^2 = 0$$

$f(\phi)$

Φ eign

$$\frac{d^2 \Phi}{d\ell^2} = -m^2 \Phi$$

$$\underline{\Phi} =$$

$$A \sin m\ell + B \cos m\ell$$

$$A e^{\pm i(m\ell + \delta)}$$



$$\underline{e}^{i\delta} = \cos \delta + i \sin \delta$$

complex: No complex voltage
take real part but use complex notation

Physics says $\underline{\Phi}(\ell) = \underline{\Phi}(\ell + 2\pi)$

$$A e^{\pm im\ell} = A e^{\pm i m (\ell + 2\pi)} = A \overline{e}^{\pm im\ell} e^{im2\pi}$$



$$m = 0, 1, 2, \dots$$

an integer

Griffiths assumes $m = 0$ azimuthal symmetry
 $\underline{\Phi}$ is a constant

$$\frac{\sin^2 \theta}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{\sin \theta}{\theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) = 0$$

\Rightarrow ODE in $R \& \Theta$ $\underline{\Theta} = \text{const}$

$V = R(r) \Theta(\theta) \frac{P_l(\theta)}{r^l}$ This only solves specific boundary problems

\uparrow const

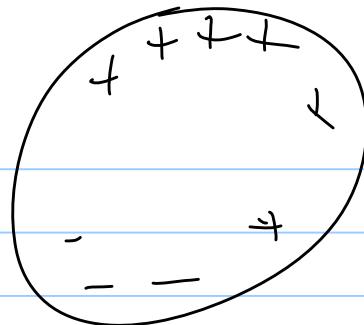
Legendre poly

for $m=0$

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)} \right) P_l(\cos \theta)$$

Superposition to get general soln

Given $V(r=R, \theta)$



find V everywhere
given V on boundary

$$\int_0^\pi P_l(\cos\theta) P_m(\cos\theta) \sin\theta d\theta = \begin{cases} 0 & \text{if } l \neq m \\ \frac{2}{2m+1} & \text{if } l = m \end{cases}$$

General soln

$$V(r, \theta, \varphi) = \sum_{l, m} \left(A_{lm} r^l + \frac{B_{lm}}{r^{l+1}} \right) Y_{lm}(\theta, \varphi)$$

$$Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi}$$

Ex: Given $V_0(\theta)$ on surface of a hollow sphere of radius R . Find V everywhere

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l \frac{1}{r^{l+1}} \right) P_l(\cos\theta)$$



$$r < R \quad B_l = 0$$

$$V = \sum A_l r^l P_l(\cos\theta)$$

2 regions

$$r > R \quad A_l = 0$$

$$V = \sum B_l \frac{1}{r^{l+1}} P_l(\cos\theta)$$

$$\text{at } r = R \quad V(R, \theta) = V_0(\theta) = \sum A_l R^l P_l(\cos\theta)$$

multiply by $P_m(\cos\theta) d(\cos\theta)$ & integrate

$$\int_{-1}^1 V_o(\theta) P_m(\cos\theta) d(\cos\theta) = \sum_{l=0}^{\infty} A_l R^l \int_{-1}^1 P_l P_m d(\cos\theta)$$

$$= \sum_{l=0}^{\infty} A_l R^l \frac{2}{2l+1} \delta_{lm} = A_m R^m \frac{2}{2m+1}$$

$$\text{So } A_l = \frac{2l+1}{2R^l} \int_{-1}^1 V_o(\theta) P_l(\cos\theta) d(\cos\theta)$$