11/20/06

Note Title 11/20/2006

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P = 0$$

axial symmetry $m=0$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \ell(\ell+1)P = 0$$

$$-2x\frac{dP}{dx} + (1-x^2)\frac{d^2P}{dx^2} + \ell(\ell+1)P = 0$$

$$P = \sum_{n=0}^{\infty} a_n x^n$$

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

$$P' = \sum_{N=1}^{\infty} N a_N x^{N-1}$$

$$x P' = \sum_{N=1}^{\infty} N a_N x^N = \sum_{N=0}^{\infty} N a_N x^N$$

$$P'' = \sum_{N=0}^{\infty} N(N-1) a_N x^{N-2}$$

$$= \sum_{N=0}^{\infty} (N+2)(N+1) a_{N+2} x^N$$

$$x^2 P'' = \sum_{N=0}^{\infty} N(N-1) a_N x^N$$

$$\boxed{(1-x^2)P'' - 2xP' + \ell(\ell+1)P = 0}$$

$$\sum_{N=0}^{\infty} \Big[(N+2)(N+1) a_{N+2} - \underline{N(N-1) a_N} - \underline{2 N a_N} + \ell(\ell+1) a_N \Big] x^N = 0$$

$$-n^2 + n - 2n = -n^2 - n$$
$$= -n(n+1)$$

recursion relation in [ ]
becomes

$$(n+2)(n+1)a_{n+2} - \left[n(n+1) - \ell(\ell+1)\right]a_n = 0$$

$$\Rightarrow \quad a_{n+2} = -\frac{\left[n(n+1) - \ell(\ell+1)\right]}{(n+2)(n+1)}\, a_n$$

in General, this gives
2 infinite series (just
as for sin, cos).

But suppose ℓ is an
integer

Then when n hits ℓ
we have

$$a_{\ell+2} = \frac{-[\ell(\ell+1) - (\ell(\ell+1))]}{(\ell+2)(\ell+1)} a_\ell$$

$$= 0$$

This means that the infinite series terminates.

i.e. for $\ell =$ integer, the solutions are polynomials of order ℓ.

$P_0(x) = 1$ $\qquad$ $P_1(x) = x$

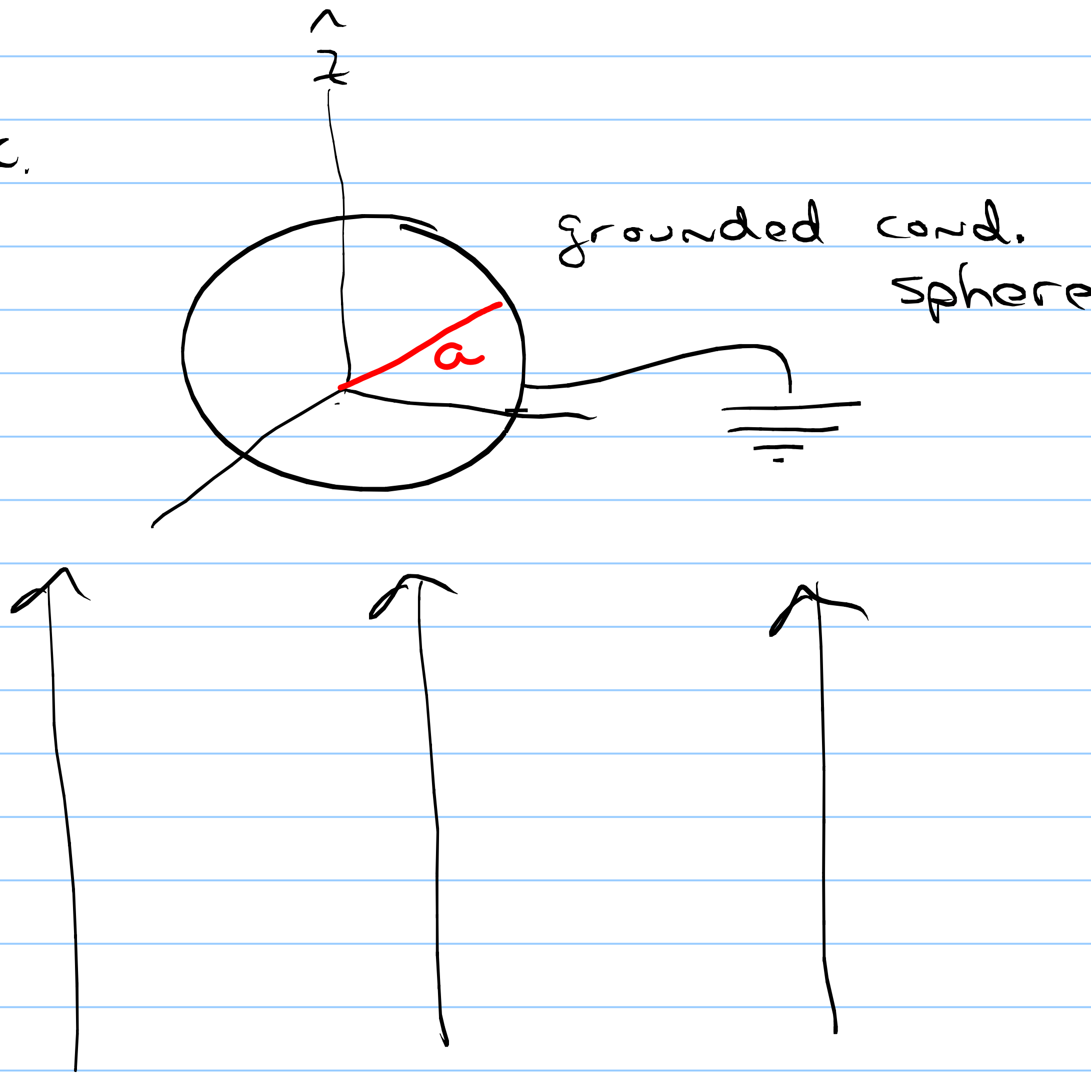
$P_2(x) = \frac{1}{2}(3x^2 - 1)$ $\qquad$ $P_3(x) = \frac{1}{2}(5x^2 - 3x)$

$\vdots$

These polynomials are orthogonal, but not normalized

<u>Mathematica</u>

Ex.



$$\vec{E} = E_0 \hat{z}$$

Axial symmetry $\Rightarrow m=0$

So our solution to $\nabla^2 V = 0$

remember $E = -\nabla V$

so $V = -E_0 z$

$$V(r,\theta,\phi) = \sum (A_{\ell m} r^{\ell} + B_{\ell m} r^{-(\ell+1)}) Y_{\ell m}(\theta,\phi)$$

$$\Downarrow$$

$$V(r,\theta) = \sum_{\ell=0}^{\infty} (A_\ell r^{\ell} + B_\ell r^{-(\ell+1)}) P_\ell(\cos\theta)$$

Since sphere is conducting

BC ① $V(r=a,\theta) = 0$

$$= \sum_{\ell=0}^{\infty} (A_\ell a^{\ell} + B_\ell a^{-(\ell+1)}) P_\ell = 0$$

BC ② Since $\bar{E}$ as $r \to \infty = E_0 \hat{z}$

$$\Rightarrow V(r\to\infty,\theta) = -E_0 z$$

BC 1 multiply both sides by $P_{\ell'}(\cos\theta)$ integrate

$$\sum_{\ell=0}^{\infty} \left(A_\ell a^\ell + B_\ell a^{-(\ell+1)}\right) \underbrace{\int_{-1}^{1} P_\ell P_{\ell'}\, dx}_{\frac{2}{2\ell+1}\delta_{\ell\ell'}}$$

$$\Rightarrow \quad A_\ell a^\ell + B_\ell a^{-(\ell+1)} = 0$$

$$\Rightarrow \quad B_\ell = -\frac{a^\ell}{a^{-(\ell+1)}} A_\ell = -a^{2\ell+1} A_\ell$$

Now B.C. 2 which applies as $r \to \infty$

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)$$

note that for large r only the A_l terms survive so a $r \to \infty$ B.C. does not constrain B_l

So

$$\lim_{r\to\infty} V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$= -E_0 z$$

$$= -E_0 r \cos\theta$$

$P_1(x) = x$ or $P_1(\cos\theta) = \cos\theta$

So $-E_0 z = -E_0 r \cos\theta$

so only the $\ell = 1$ term is involved

$$\lim_{r \to \infty} V(r,\theta) = A_1 r P_1(\cos\theta)$$
$$= -E_0 r P_1(\cos\theta)$$

$$\Rightarrow \boxed{A_1 = -E_0}$$

BC 1 said

$$B_\ell = -a^{2\ell+1} A_\ell$$
$$\Rightarrow B_1 = -a^3(-E_0)$$

Putting this all together

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right) P_\ell$$

$$= \left(A_1 r + B_1 r^{-2}\right) P_1$$

$$= \left(-E_0 r + a^3 E_0 r^{-2}\right) \cos\theta$$

$$= -E_0 \left(1 - \frac{a^3}{r^3}\right) r \cos\theta$$

Exercise: compute ∇ is spherical coord

You'll get

$$E_r = E_0 \left(1 + 2\left(\frac{a}{r}\right)^3\right) \cos\theta$$

$$E_\theta = -E_0 \left(1 - \left(\frac{a}{r}\right)^3\right) \sin\theta$$

check out

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