

Gaussian beams:

sketch of method -

$$u(x, y) = E_0 e^{-\frac{(x_0^2 + y_0^2)}{w_0^2}}$$

Fresnel integral to propagate forward:

$$u(x, y, z) = \frac{i E_0}{\lambda z} e^{-ikz} \iint e^{-\frac{(x_0^2 + y_0^2)}{w_0^2} \left(\frac{1}{w_0^2} + \frac{ik}{2z} \right) + i(\beta_x x_0 + \beta_y y_0)} dx_0 dy_0$$

so this is a F.T. of a Gaussian (complex) which will stay Gaussian in shape.

define "q" parameter:

$$\frac{1}{w_0^2} + \frac{ik}{2z} = \frac{ik}{2} \left(\frac{-2i}{kw_0^2} + \frac{1}{z} \right) \equiv \frac{ik}{2q}$$

$$\text{with } \frac{1}{q} \equiv \left(\frac{-i\lambda}{\pi w_0^2} + \frac{1}{z} \right)$$

Svelto, Siegman
+ int convention

$$u(x, y, z) = \frac{i E_0}{\lambda z} e^{-ikz} \iint e^{-\frac{ik}{2q} (x_0^2 + y_0^2) + i(\beta_x x_0 + \beta_y y_0)} dx_0 dy_0$$

if we treat the beam as a Gaussian with an effective size

$$w_{\text{eff}}^2 \equiv \frac{2q}{ik}$$

then we know qualitatively that $\text{FT}\{\text{gaussian}\} \rightarrow \text{gaussian}$.

$$u(x, y, z) = \frac{i E_0}{\lambda z} e^{-ikz} \left(\frac{e^{+i q \beta_x^2 / 4k} \sqrt{\pi}}{\sqrt{+ik/q}} \right) \left(\frac{e^{+i q \beta_y^2 / 4k} \sqrt{\pi}}{\sqrt{+ik/q}} \right)$$

$$= \frac{E_0}{2z/q} e^{-ikz} e^{i \frac{kq}{4z} (x^2 + y^2)}$$

gaussian form.

+i\omega t

Final form:

$$u(x, y, z) = \frac{w_0}{w} e^{-\left(\frac{x^2+y^2}{w^2}\right)} e^{-ik\left(\frac{x^2+y^2}{2R}\right)} e^{i\phi}$$

$$\text{with } w^2(z) = w_0^2 \left(1 + \frac{z^2}{z_R^2}\right)^2$$

$$z_R = \frac{\pi w_0^2}{\lambda}$$

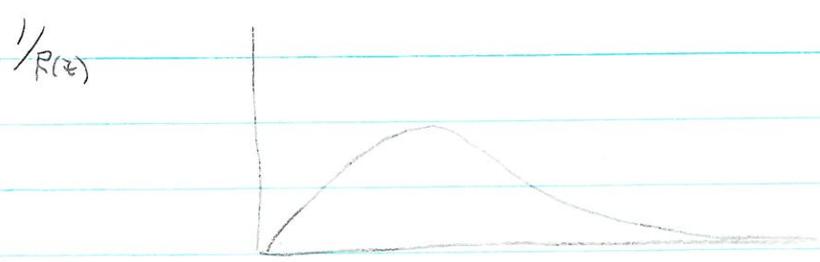
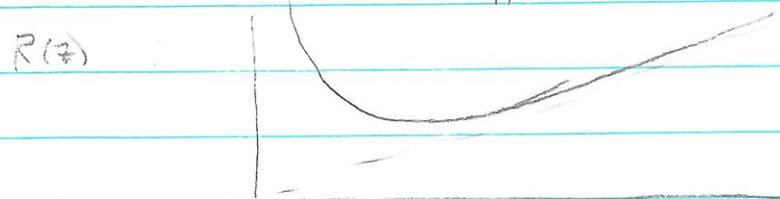
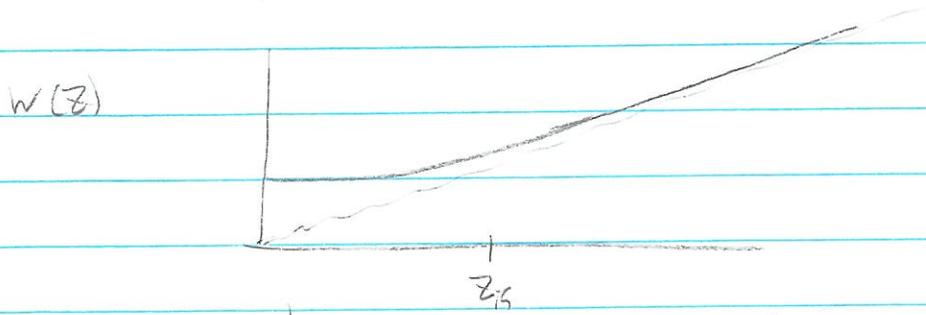
$$R(z) = z \left(1 + \frac{z_R^2}{z^2}\right)$$

= Rayleigh length

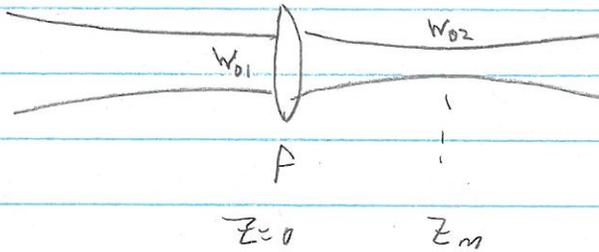
$$\phi(z) = \tan^{-1}\left(\frac{z}{z_R}\right)$$

$$\frac{1}{q} = \frac{1}{R} - i \left(\frac{\lambda}{\pi w^2}\right)$$

$$q = \frac{Aq_1 + B}{Cq_1 + D}$$



Focusing a Gaussian beam.



Z_m = distance to waist
 $\neq f$ in general

$$w_{01} = w_{02} \sqrt{1 + \frac{Z_m^2}{Z_{R2}^2}} \quad \rightarrow \quad Z_{R1} = Z_{R2} \left(1 + \frac{Z_m^2}{Z_{R2}^2}\right)$$

at lens $R = f = Z_m \left(1 + \frac{Z_{R2}^2}{Z_m^2}\right)$

Find Z_m (dist to focus)

$$Z_{R2} = \frac{\pi w_{02}^2}{\lambda}$$

divide equations: $\frac{f}{Z_{R1}} = \frac{Z_m \left(1 + \frac{Z_{R2}^2}{Z_m^2}\right)}{Z_{R2} \left(1 + \frac{Z_m^2}{Z_{R2}^2}\right)}$

let $X_2 = Z_m / Z_{R2} \rightarrow X_1 = X_2 \frac{1 + 1/X_2^2}{1 + X_2^2}$
 $X_1 = f / Z_{R1}$

$$(1 + X_2^2) X_2^2 X_1 = X_2 (X_2^2 + 1)$$

$$\rightarrow X_1 = 1/X_2 \quad \text{or} \quad \frac{f}{Z_{R1}} = \frac{Z_{R2}}{Z_m}$$

$$\rightarrow Z_{R1} = Z_{R2} \left(1 + \frac{Z_{R1}^2}{f^2}\right) \rightarrow \boxed{Z_{R2} = \frac{Z_{R1}}{1 + \frac{Z_{R1}^2}{f^2}}}$$

$$f = Z_m \left(1 + \frac{f^2}{Z_{R1}^2}\right)$$

$$\rightarrow \boxed{Z_m = \frac{f}{1 + \frac{f^2}{Z_{R1}^2}}}$$

$Z_m < f$ b/c of diffraction.

calculate focused spot size:

$$W_{02}^2 = \frac{W_{01}^2}{1 + z_{R1}^2/f^2}$$

when $z_{R1}/f \gg 1$ diffraction does not dominate

$$W_{02}^2 \approx \frac{W_{01}^2 f^2 / z_{R1}^2}{1 + f^2 / z_{R1}^2} = \frac{f^2 \lambda^2}{\pi^2 W_{01}^2} \frac{1}{1 + f^2 / z_{R1}^2}$$

in the limit $z_{R1} \gg f$

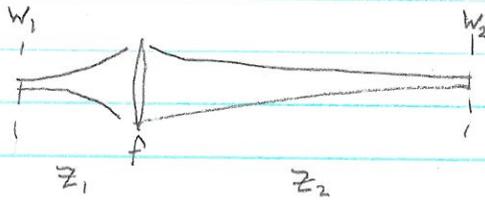
$$W_{02} = \frac{1}{\pi} \lambda \frac{f}{W_{01}} = \frac{\lambda}{\pi} \frac{f}{(2W_{01})}$$

$$f / (2W_{01}) = f\text{-number of focusing} = f/\#$$

this is what you get by taking Fourier transform of input Gaussian

Gaussian beam exercise

start with beam waist \rightarrow diverge \rightarrow focus w/ lens \rightarrow waist



fixed $z_1 + z_2$
find z_1 to have a
waist at z_2

2 ways: Gaussian beam formulas
ABCD

Gaussian beam:

at lens: $w_2 = w_1 \sqrt{1 + \frac{z_1^2}{z_{R1}^2}}$

$$R_1 = z_1 \left(1 + \frac{z_{R1}^2}{z_1^2} \right)$$

after lens

$$w = w_0$$

$$\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f}$$

$$R_2 = -z_2 \left(1 + \frac{z_{R2}^2}{z_2^2} \right)$$

$$w_2 = w_0 \left(1 + \frac{z_2^2}{z_{R2}^2} \right)$$

$\left. \begin{array}{l} R > 0 \\ R < 0 \end{array} \right\}$

ABCD method:

$$\begin{aligned} \text{system matrix: } & \begin{pmatrix} 1 & z_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & z_1 \\ 0 & 1 \end{pmatrix} \\ & \begin{pmatrix} 1 & z_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & z_1 \\ -1/f & 1 - z_1/f \end{pmatrix} \\ & \begin{pmatrix} 1 - z_2/f & z_1 + z_2(1 - z_1/f) \\ -1/f & 1 - z_1/f \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{aligned}$$

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} = \frac{(1 - z_2/f)q_1 + z_1 + z_2(1 - z_1/f)}{-q_1/f + 1 - z_1/f}$$