MATH 332 - Linear Algebra July 16, 2009

Homework 7, Summer 2009 **Due**: July 24, 2009

Dimension - Rank - Eigenproblems - Markov Chains - Dynamical Systems

1. Determine the eigenvalues and a basis for the eigenspace of $\mathbf{A}$ given by,

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

2. Given,

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

Is $\mathbf{A}$ diagonalizable? If so, then determine $\mathbf{D}$ and $\mathbf{P}$ associated with a diagonal decomposition $\mathbf{PDP}^{-1}$ of $\mathbf{A}$.

3. Prove the following statements:

   (a) dim Row $\mathbf{A}$ + dim Nul $\mathbf{A} = n$ where $\mathbf{A} \in \mathbb{R}^{m\times n}$.

   (b) Rank $\mathbf{A}$ + dim Nul $\mathbf{A}^{\mathrm{T}} = m$ where $\mathbf{A} \in \mathbb{R}^{m\times n}$.

   (c) $\mathbf{Ax}$=$\mathbf{b}$ has a solution for each $\mathbf{b}\in \mathbb{R}^m$ if and only if the equation $\mathbf{A}^{\mathrm{T}}\mathbf{x} = \mathbf{0}$ has only the trivial solution.[1]

   (d) The characteristic polynomial of $\mathbf{A}$ is equal to the characteristic polynomial of $\mathbf{A}^{\mathrm{T}}$. [2]

   (e) If $\mathbf{A}$ is an invertible matrix with eigenvalue λ then λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$.[3]

   (f) If $\mathbf{A}$ is both diagonalizable and invertible, then so is $\mathbf{A}^{-1}$.[4]

   (g) If $\mathbf{A}$ has n linearly independent eigenvectors, then so does $\mathbf{A}^{\mathrm{T}}$. [5]

4. Square matrices having columns whose entries sum to 1 are often called stochastic matrices. Those with only non-negative entries, for some power, are called *regular* stochastic matrices. Given a random process, with an initial state $\mathbf{x}_0$, the application of $\mathbf{P}$ on $\mathbf{x}_0$ discretely steps the process forward in time. That is $\mathbf{x}_{n+1} = \mathbf{P}\mathbf{x}_n = \mathbf{P}^n\mathbf{x}_0$, $n = 1, 2, 3, \ldots$. If a matrix is a *regular* stochastic matrix then there exists a steady-state vector $\mathbf{q}$ such that $\mathbf{Pq}$=$\mathbf{q}$. This vector determines the long term probabilities associated with an arbitrary inital state $\mathbf{x}_0$. The sequence of states, $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n+1}\}$, is called a *Markov Chain.* Given the regular stochastic matrix:

$$\mathbf{P} = \begin{bmatrix} .1 & .6 \\ .9 & .4 \end{bmatrix}.$$

   (a) Show that the steady-state vector of $\mathbf{P}$ is $\mathbf{q} = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \end{bmatrix}^{\mathrm{T}}$. [6]

   (b) Find the matrices $\mathbf{D}$ and $\mathbf{Q}$ such that $\mathbf{P}$= $\mathbf{QDQ}^{-1}$. That is, diagonalize the matrix $\mathbf{P}$.

   (c) Show that $\lim\limits_{n\to\infty} \mathbf{P}^n\mathbf{x}_0 = \mathbf{q}$ where $\mathbf{x}_0 = [x_1,\, x_2]^{\mathrm{T}}$ is an arbitrary vector in $\mathbb{R}^2$ such that $x_1 + x_2 = 1$. [7]

5. Given,

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}.$$

Determine the eigenvalues and eigenfunctions associated with the system of differential equations $\frac{d\mathbf{x}}{dt} = \mathbf{A}\cdot\mathbf{x}(t)$.

[1]For the forward direction use theorem 1.4.4 on page 43 and problem 3b to prove that the dimension of the null space of $\mathbf{A}^{\mathrm{T}}$ is zero.

[2]Note that $\mathbf{I}$ is a symmetric matrix then use rules for the transposition of a sum and determinants of transposes.

[3]Start with $\mathbf{Ax} = \lambda\mathbf{x}$ and multiply on the left by $\mathbf{A}^{-1}$.

[4]Note that if $\mathbf{D}$ is a diagonal matrix then $\mathbf{D}^{-1}$ is the matrix whose diagonal elements are scalar inverses of the diagonal elements of $\mathbf{D}$.

[5]Use theorem 5.3.5 and the fact that if $\mathbf{P}$ is invertible then $(\mathbf{P}^{\mathrm{T}})^{-1} = (\mathbf{P}^{-1})^{\mathrm{T}}$. It is also useful to note that diagonal matrices are symmetric.

[6]You could try to solve $\mathbf{Px} = \mathbf{x}$ for $\mathbf{x}$, but is easier to show that $\mathbf{Pq} = \mathbf{q}$.

[7]Notice that $\mathbf{P}^n = \mathbf{QD}^n\mathbf{Q}^{-1}$ allows you to replace $\lim\limits_{n\to\infty} \mathbf{P}^n = \mathbf{Q} \lim\limits_{n\to\infty} \mathbf{D}^n\ \mathbf{Q}^{-1}$, where the limit can now be calculated.