MATH348: LINEAR PDE WITH PERIODIC BOUNDARY CONDITIONS

All I know is that to me you look like you're lots of fun.

1. Introduction

Notice that the following equation defines an eigenvalue problem,

$$\frac{d^2}{dx^2}X = -\lambda X, \tag{1}$$

where $-\lambda$ is the eigenvalue and X is the *eigenfunction*. From our boundary value problems we found the following eigenvalue/eigenfunction pairs.

$$X(0) = 0, \quad X(L) = 0 \implies X_n = \sin(\sqrt{\lambda_n}x), \quad \lambda_n = \frac{n^2\pi^2}{L}, \; n = 1, 2, 3, \ldots,$$

$$X'(0) = 0, \quad X'(L) = 0 \implies X_n = \cos(\sqrt{\lambda_n}x), \quad \lambda_n = \frac{n^2\pi^2}{L}, \; n = 0, 1, 2 \ldots,$$

$$X(0) = 0, \quad X'(L) = 0 \implies X_n = \sin(\sqrt{\lambda_n}x), \quad \lambda_n = \frac{(2n-1)^2\pi^L}{2L}, \; n = 1, 2, 3, \ldots,$$

$$X'(0) = 0, \quad X(L) = 0 \implies X_n = \cos(\sqrt{\lambda_n}x), \quad \lambda_n = \frac{(2n-1)^2\pi^L}{2L}, \; n = 1, 2, 3, \ldots,$$

If instead of these boundary conditions we might consider the following for a $2L-$unit object,

$$X(-L) = 0, \; X(L) = 0, \text{ and } X'(L) = 0, \; X'(L) = 0, \implies$$

$$\implies X_n = \sin(\sqrt{\lambda_n}x), \text{ and } X_n = \cos(\sqrt{\lambda_n}x), \lambda_n = \frac{n^2\pi^2}{L}, \; n = 0, 1, 2, \ldots,$$

which says that the solution to the eigenproblem must be a periodic function. Physically one could think of this as heat flow on a ring. Mathematically, we have more than one function that solves the problem for each non-zero eigenvalue, which is to say that each non-zero eigenvalue is repeated once. Regardless, the matter of importance here is that these functions obey the following integrals,

$$\int_{-a}^{b} \cos\left(\sqrt{\lambda_n}x\right) \cos\left(\sqrt{\lambda_m}x\right) dx = L\delta_{nm}, \tag{2}$$

$$\int_{-a}^{b} \sin\left(\sqrt{\lambda_n}x\right) \sin\left(\sqrt{\lambda_m}x\right) dx = L\delta_{nm}, \tag{3}$$

$$\int_{-a}^{b} \sin\left(\sqrt{\lambda_n}x\right) \cos\left(\sqrt{\lambda_m}x\right) dx = 0, \tag{4}$$

where $b - a = 2L$ and $\sqrt{\lambda_n} = n\pi/L$ such that $n = 1, 2, 3, \ldots$. We take this to mean that the eigenfunctions of the problem satisfy *orthogonality conditions*. To see what we mean by this, we consider the following problem.

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2. Heat Equation on a Lattice

The initial-boundary value problem,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad x \in (0, L), \quad t \in (0, \infty), \tag{5}$$

$$u(-L, t) = u(L, t), \quad u_x(-L, t) = u_x(L, t), \tag{6}$$

$$u(x, 0) = f(x). \tag{7}$$

is known as the diffusion equation and models the flow of a conserved density on a ring with circumference $2L$. The constant c^2 is known as the diffusivity constant and contains the material properties of a conserved density that must obey the second-law of thermodynamics. The diffusivity constant measures how easily the material permits stuff to flow through it. If u is a thermal energy density, temperature, then we call this the heat equation. In this case, the boundary conditions say that the temperature on the left must be equivalent to that on the right. So, we could think of allowing heat to flow on a circle and asking what are the dynamics of the temperature on the circle. Regardless, we treat this through separation of variables to get,

$$u(x, t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\sqrt{\lambda_n} x\right) e^{-\lambda_n c^2 t} + b_n \sin\left(\sqrt{\lambda_n} x\right) e^{-\lambda_n c^2 t} \tag{8}$$

Now, if we state the solution at the initial time we find,

$$a_0 = \frac{1}{2L} \int_{-L}^{L} u(x, 0) dx \tag{9}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} u(x, 0) \cos\left(\sqrt{\lambda_n} x\right) dx \tag{10}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} u(x, 0) \sin\left(\sqrt{\lambda_n} x\right) dx \tag{11}$$

(12)

These Fourier coefficients define the shape of the temperature at time zero and the heat equation evolves this shape through the Fourier series solution Eq. (8). Mathematically we could say that at each point in time the solution to the PDE exists in an infinite-dimensional phase space, whose basis vectors are the orthogonal trigonometric functions. We again note the following interesting limit,

$$\lim_{t \to \infty} u(x, t) = u_{\text{avg}}(x, 0). \tag{13}$$

3. Conclusions

My point here is that the predicted dynamics are pretty simple and expected of our physical intuition of how heat flow should behave. The difficulty is in understanding the meaning of the series solution for any point in time. That is, we care to understand the meaning of

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\sqrt{\lambda_n} x\right) + b_n \sin\left(\sqrt{\lambda_n} x\right), \tag{14}$$

which is known as a Fourier series. Noting the following formulae,

$$2\cos(x) = e^{ix} + e^{-ix}, \tag{15}$$

$$2i\sin(x) = e^{ix} - e^{-ix}, \tag{16}$$

which follows from Euler's formulae, allows us to re-write the *real* Fourier series above in a *complex* form,

$$f(x) = \sum_{n=-\infty}^{\infty} c\left(\sqrt{\lambda_n}\right) e^{i\sqrt{\lambda_n}x}, \tag{17}$$

$$c\left(\sqrt{\lambda_n}\right) = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\sqrt{\lambda_n}x}dx. \tag{18}$$

This is the form we will study. The real form is particularly useful when working with PDE but when not considering PDE, the compactness of the complex form shines.

4. Things to Do

This PDE brings together many of the ideas from our previous work and centers our attention on the underlying concept of a series of trigonometric functions, known as a Fourier series. The following list are action items for the above discussion.

1. Derive Eq. (8) from Eq. (5) – (7).
2. Using Eq. (2)–(4), derive Eq. (9)–(11).
3. Using Eq. (15)–(16), derive Eq. (17)–(18) from Eq. (17) and (9)–(11).

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