Sample probs for final part 1

Note Title 12/3/2007

Boas 13 section 10

Prob. 9 cf Lecture notes for 11/9/07

Solution to the string problem

$$u(x,t) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{\omega_n}{c}x\right)\cos(\omega_n t) + B_n \sin\left(\frac{\omega_n}{c}x\right)\sin(\omega_n t)$$

$$u(x, t=0) = \sum A_n \sin\left(\frac{\omega_n}{c}x\right)$$

Assume $\frac{\partial u}{\partial t} = 0$ at $t=0$. This implies that all the B coef. are 0.

So, Find A_n such that

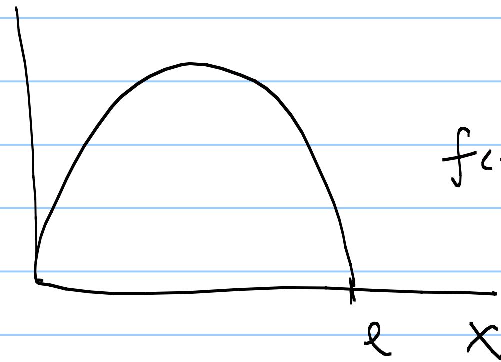
$$\sum_{n=0}^{\infty} \underline{\underline{A_n}} \sin\left(\frac{\omega_n}{c}x\right) = x(\ell - x)$$

Then plug these into

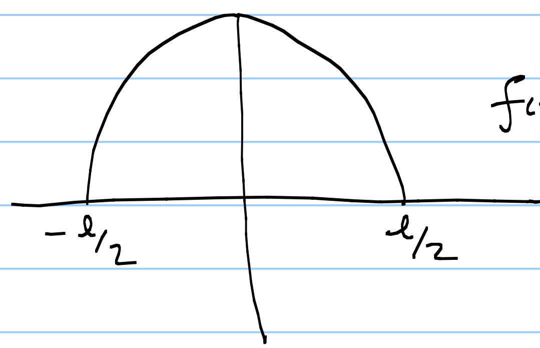
$$u(x,t) = \sum A_n \sin\left(\frac{\omega_n}{c}x\right)\cos(\omega_n t)$$

That will be the motion of the string after you let go.

You can use mathematica to get the A_N. To do so note the following



$$f(x) = x(\ell - x)$$



$$\underline{f(x) = \left(x+\tfrac{\ell}{2}\right)\left(\ell/2 - x\right)}$$

This is what mathematica wants

E.g. for $\ell = 1$

$$f(x) = \frac{1}{6} + \frac{1}{\pi^2}\cos(2\pi x) - \frac{1}{4\pi^2}\cos(4\pi x) \ldots$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$A_0/2 \qquad A_2 \qquad A_4 \qquad \ldots$$

$$\boxed{A_{2N} = (-1)^{N+1}\frac{1}{N^2}\frac{1}{\pi^2}} \qquad N = 1, 2, 3$$

$$A_{2N-1} = 0$$

$$u(x,t) = \sum_N A_N \sin\left(\frac{\omega_N}{c}x\right)\cos(\omega_N t)$$

Klein-Gorden in 1D

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \lambda^2 u$$

$\downarrow$ $\frac{u}{\text{dist}^2}$ $\quad\quad$ $\downarrow$ $\frac{u}{\frac{\text{dist}^2}{\text{time}^2}\,\text{time}^2}$ $\quad\quad$ must be $\frac{1}{\text{dist}^2}$

$\frac{1}{\text{time}}$ = frequency $\quad\quad$ $\frac{1}{\text{distance}}$ = wavenumber

S.O.V.

$$u(x,t) = X(x)T(t)$$

① $c^2 X'' T = X\ddot{T} + c^2\lambda^2 XT$

$\uparrow$ this will make the sep. const a frequency

② $c^2 \frac{X''}{X} = \frac{\ddot{T}}{T} + c^2\lambda^2 = -\omega^2$

$\Rightarrow \quad X'' + \frac{\omega^2}{c^2} X = 0 \quad\quad$ ODE 1

$\ddot{T} + (\omega^2 + c^2\lambda^2)T = 0 \quad\quad$ ODE 2

Apply clamped end BC

$$X(x) = A \sin\left(\frac{\omega}{c}x\right)$$

$$\frac{\omega}{c}L = n\pi \Rightarrow \omega_n = \frac{n\pi c}{L}$$

$$\ddot{T} + \left(\frac{n^2\pi^2c^2}{L^2} + c^2\lambda^2\right)T = 0$$

i.e. $\ddot{T} + [\;]T = 0$ Harmonic osc.

└ must be $(2\pi f)^2$

$$(2\pi f_n)^2 = \pi^2c^2\left[\frac{n^2}{L^2} + \frac{\lambda^2}{\pi^2}\right]$$

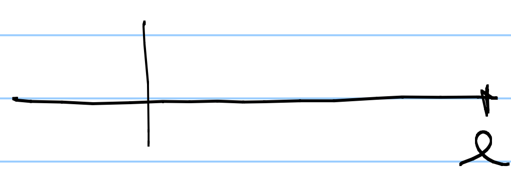
$$f_n^2 = \left(\frac{c}{2}\right)^2\left(\frac{n^2}{L^2} + \frac{\lambda^2}{\pi^2}\right)$$

$$\boxed{f_n = \frac{c}{2}\sqrt{\frac{n^2}{L^2} + \lambda^2/\pi^2}}$$

In steady state Diffusion $\Rightarrow \nabla^2 T = 0$

Examples

1D Bar



$T(0) = T_0$

$T(\ell) = T_1$

$$\nabla^2 T = \frac{d^2T}{dx^2} = 0$$

$$\Rightarrow T = ax + b$$

$$T(0) = b = T_0$$

$$T(\ell) = a\ell + T_0 = T_1$$

$$\Rightarrow a = \frac{T_1 - T_0}{\ell}$$

$$\boxed{T(x) = \left(\frac{T_1 - T_0}{\ell}\right)x + T_0}$$

Problem. Apply Sep. of Var. to reduce $\nabla^2 \psi(r,\theta) = 0$ to two ODE's but don't solve them

2D cylindrical coordinates $\nabla^2 \psi(r,\theta) = 0$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0$$

S.O.V. $\psi = R(r)P(\theta)$

$$\frac{1}{r}\frac{d}{dr}(rR'P) + \frac{1}{r^2}RP'' = 0$$

$$\frac{1}{r}\left[R'P + rR''P\right] + \frac{1}{r^2}RP'' = 0$$

$\times\, r^2$

$$r^2R''P + rR'P + RP'' = 0$$

$$\Rightarrow r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{P''}{P} = \alpha^2$$

$$P'' + \alpha^2 P = 0 \quad \Rightarrow \quad P \sim e^{i\alpha\theta}$$

$$\boxed{R'' + \frac{1}{r}R' - \frac{\alpha^2}{r^2}R = 0}$$

Example of simplifying

as $r \to 0$

$$\frac{1}{r} R' - \frac{\alpha^2}{r^2} R = 0$$

$$R' - \frac{\alpha^2}{r} R = 0$$

$$\Rightarrow \quad R' = \frac{\alpha^2}{r} R$$

$$\Rightarrow \quad \frac{dR}{R} = \alpha^2 \frac{dr}{r}$$

$$\ln R \approx \alpha^2 \ln r$$

$$R \approx e^{\alpha^2 \ln r} = r^{\alpha^2}$$

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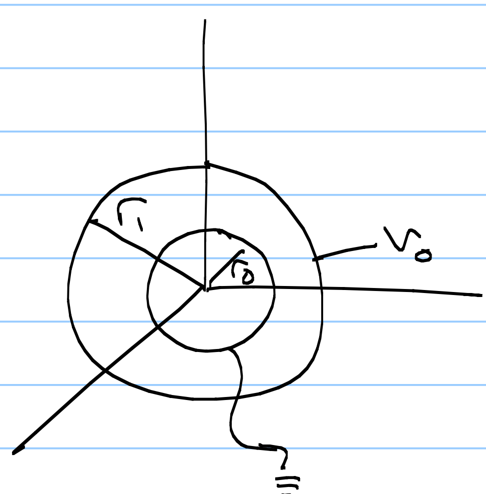
7 in notes



Find potential between spheres

$$\nabla^2 V = 0$$

$$V(r=r_0) = 0$$

$$V(r=r_1) = V_0$$

spherical symmetry $V = V(r)$

$$\nabla^2 V(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

$$\Rightarrow \left(r^2 \frac{\partial V}{\partial r}\right) = \text{constant} \equiv C_0$$

$$\frac{\partial V}{\partial r} = \frac{C_0}{r^2}$$

$$\Rightarrow V(r) = \frac{-C_0}{r} + C_1$$

$$V(r_0) = 0 = -\frac{C_0}{r_0} + C_1 \Rightarrow C_1 = \frac{C_0}{r_0}$$

$$V(r_1) = V_0 = -\frac{C_0}{r_1} + C_1$$

$$= -\frac{C_0}{r_1} + \frac{C_0}{r_0} = C_0\left[\frac{1}{r_0} - \frac{1}{r_1}\right]$$

$$\Rightarrow C_0 = \frac{V_0}{\frac{1}{r_0} - \frac{1}{r_1}}$$

$$\Rightarrow C_1 = \frac{V_0}{1 - r_0/r_1} = \frac{r_1}{r_1 - r_0} V_0$$

$$V(r) = V_0 \frac{r_1}{r}\left(\frac{r - r_0}{r - r_1}\right) = V_0 \left(\frac{1 - r_0/r}{1 - r_0/r_1}\right)$$

Always check

$$V(r_0) = \frac{V_0 \left(1 - r_0/r_0\right)}{1 - r_0/r_1} = 0$$

$$V(r_1) = V_0 \left(\frac{1 - r_0/r_1}{1 - r_0/r_1}\right) = V_0$$

Orthogonal Polynom. example

given
$$P_0 = 1$$
$$P_1 = x$$
$$P_2 = \tfrac{1}{2}(3x^2 - 1)$$
on $[-1, 1]$

find $P_3 = ax^3 + bx^2 + cx + d$

such that $(P_i, P_j) = 0$ if $i \neq j$

$$(P_0, P_3) = \int_{-1}^{1} ax^3 + bx^2 + cx + d \, dx$$

$$\boxed{\tfrac{2}{3}b + 2d = 0}$$

$$(P_3, P_1) = \boxed{\frac{2}{5}a + \frac{2}{3}c = 0}$$

$$(P_3, P_2) = \boxed{\frac{4b}{15} = 0}$$

$\Rightarrow b = 0 \Rightarrow d = 0$

This leaves $a = -5/3\, c$

Let $c = 1$ then [4 unknowns, 3 equations

$$P_3(x) = -\frac{5}{3}x^3 + x$$

$$\boxed{P_3(x) = 5x^3 - 3x}$$

You could use the normalization $(P_\ell, P_\ell) = \frac{2}{2\ell+1}$ to uniquely specify the 4th constant, but I would give you this.