Split-step beam propagation

concept:

- linear propagation can be calculated easily when the beam is represented in the angular spectrum
- Any variation in the spatial distribution of the refractive index must be applied in the space domain.
    - includes both static and nonlinear changes to $n(x)$

start w/ wave eqn:

$$\nabla^2 U + k^2 U = 0 \qquad k^2 = n^2 \omega^2 / c^2$$

for now, $n(\vec{r})$, ω constant

represent U in fourier space (angular spectrum)
- transverse only

$$U(x,y,z) = \mathcal{F}^{-1}\{A(f_x, f_y; z)\}$$

$$= \iint A(f_x, f_y; z)\, e^{i 2\pi (f_x x + f_y y)}\, df_x\, df_y$$

also, $$A(f_x, f_y; z) = \iint U(x,y,z)\, e^{-i 2\pi (f_x x + f_y y)}\, dx\, dy$$

no 2π out front, opposite sign convention originates from e^{+ikz} convention.

f_x, f_y are spatial frequencies

plane wave: $U_k(x,y,z;t) = e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

drop time dependence

$\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z$

can write $k_x = k\cos\theta_x \equiv k\alpha$

$k_y = k\cos\theta_y \equiv k\beta$

$k_z = k\cos\theta_z \equiv k\gamma$

α, β, γ are "direction cosines"

since $k_x^2 + k_y^2 + k_z^2 = k^2$ $\quad (k = n k_0 = n(\omega)\omega/c)$

$\alpha^2 + \beta^2 + \gamma^2 = 1$

Now we can write plane wave as

$$U_k = e^{i\frac{2\pi}{\lambda}(\alpha x + \beta y)} e^{i\frac{2\pi}{\lambda}\gamma z}$$

$$\mathcal{F}\left\{e^{i\frac{2\pi}{\lambda}\alpha x}\right\} = \int e^{i2\pi\left(\frac{\alpha}{\lambda} - f_x\right)x} dx$$
$$= \delta(f_x - \alpha/\lambda)$$

$\therefore$ identify $f_x = \alpha/\lambda \quad f_y = \beta/\lambda$

- spatial freq. have units of m^{-1}
- for fixed λ $A(f_x, f_y)$ corresponds to a distribution of plane waves traveling at angles α, β $\therefore$ ang. spectrum.

Back to wave eqn:

$$(\partial_x^2 + \partial_y^2 + \partial_z^2)U + k^2 U = 0$$

$$\rightarrow \partial_z^2 A(f_x, f_y; z) + \left(\frac{2\pi}{\lambda}\right)^2(1 - \alpha^2 - \beta^2)A(f_x, f_y; z) = 0$$

solution: $A(f_x, f_y; z) = A(f_x, f_y; 0)\, e^{i\frac{2\pi}{\lambda}\sqrt{1-\alpha^2-\beta^2}\, z}$

- propagation gives a phase shift for each ang. component.
- if $\alpha^2 + \beta^2 > 1$ wave doesn't propagate forward.

$\therefore$ must filter spectrum before propagating.

- example: sharp edge.

∴ propagation (=diffraction) is a transfer function in freq. space:

$$H(f_x, f_y; z) = \mathrm{Exp}\left[i \frac{2\pi}{\lambda} \sqrt{1-(\lambda f_x)^2 - (\lambda f_y)^2}\, z \right] \times \mathrm{circ}\left(\sqrt{(\lambda f_x)^2 + (\lambda f_y)^2} \right)$$

$$\mathrm{circ}(\rho) = 1 \text{ for } \rho \le 1$$
$$= 0 \text{ for } \rho \ge 1$$

H is called the propagator.

back to space domain:

$$U(x_0, y_0, z) = \mathcal{F}^{-1}\left\{ A(f_x, f_y; 0)\, H(f_x, f_y; z) \right\}$$

this propagator is non-paraxial: if $\vec{k}$ components are all near $\hat{z}$, $\alpha, \beta \ll 1$

$$\sqrt{1-\alpha^2-\beta^2} \rightarrow 1 - \tfrac{1}{2}\alpha^2 - \tfrac{1}{2}\beta^2 = 1 - \tfrac{1}{2}(\lambda f_x)^2 - \tfrac{1}{2}(\lambda f_y)^2$$

Now $H(f_x, f_y; z) = \mathrm{Exp}\left[i \frac{2\pi}{\lambda} z \right] \exp\left[-i\pi\lambda (f_x^2 + f_y^2) \right]$

This is Fresnel approximation.
- note + quadratic phase factor
- analogous to second-order dispersion

can continue Taylor series, or use full propagator.

if $e^{i2\pi\phi(f_x, f_y)} = e^{i k(f_x, f_y) z}$ describes propagation

$$k(f_x, f_y)\Big|_0 = k_0 + f_x k_1 + f_y k_1 + \tfrac{1}{2} f_x^2 k_2 + \tfrac{1}{2} f_y^2 k_2 + \cdots$$

$$k_2 = \frac{\partial^2}{\partial f_x^2} k \Big|_0 = \frac{2\pi}{\lambda}(-\lambda^2) = -2\pi\lambda$$

this is the equivalent of k_2 in t/ω domain (negative!)

Propagation with spatial variations in refractive index

$$\nabla^2 U + k_0^2 n^2(x,y,z)\, U = 0$$

index may be non-linear

if we ignore diffraction over a small Δz,

$$\to \partial_z^2 U + k_0^2 n^2 U = 0$$

$$U(x,y,z) = U(x,y,0)\, e^{i k_0 n(x,y)\, z}$$

assume n is indep of z, or varies slowly.

simple transfer function in space domain

can use this to describe propagation through guided-wave structures.

- slow variation in z.

NLS (nonlinear Schrödinger) eqn. in space domain

$$U(x,y,z) = A(x,y,z)\, e^{i k_0 z}$$

$$\nabla^2 U + \frac{\omega^2}{c^2} n^2 U = -\frac{4\pi \omega^2}{c^2} P_{NL}$$

$$\nabla_T^2 A + \partial_z^2 A - k_0^2 A + 2 i k_0 \partial_z A + \frac{\omega^2}{c^2} n^2 A = -\frac{4\pi\omega^2}{c^2} P_{NL}$$

- for n = constant in space, $k_0 = \frac{\omega n}{c}$
- slow varying in z $\partial_z^2 A \to 0$
- $P_{NL} = 3\chi^{(3)} |A|^2 A$

$$\to 2 i k_0 \partial_z A + \partial_x^2 A = -\frac{12\pi\omega^2}{c^2}\chi^{(3)}|A|^2 A$$

or $$\frac{\partial A}{\partial z} - \frac{1}{2} i \frac{1}{k_0}\frac{\partial^2}{\partial x^2} A = + i\gamma |A|^2 A \qquad \gamma = \frac{6\pi\omega}{nc}\chi^{(3)}$$

compare to t-domain:

$$\frac{\partial A}{\partial z} + \frac{1}{2} i k_2 \frac{\partial^2 A}{\partial \tau^2} = i \gamma |A|^2 A$$

same sign γ

$k_2 \approx -1/k_0 = -\lambda/2\pi$

disagrees w/ expansion of $\phi(t_x, t_y)$

Comments on beam propagation method:

- forward waves in z-only.
  - > can't handle back reflections, phase conjugation.
  - > slow varying in $n(z)$.
- not vector-based:
  - boundary conditions aren't figured in
  - meant for smooth variations in $n(x,y)$.

- dispersive term comes from diffraction
  - no real way to change experimentally.