Homework #8 Solutions

1. Calculate the following Fouier sine/cosine transformations. Please include the domain which the transformations is valid.
a) $F_c\left\{e^{-ax}\right\}, a > 0, x > 0 \Rightarrow f(x) = e^{-ax}$
Thus,

$$\begin{aligned}
\hat{f}_c(w) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)cos(wx)dx = \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax}cos(wx)dx = \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+w^2}\left\{w\cdot sin(wx) + (-a)cos(wx)|_0^\infty\right\}\right] = \\
&= \sqrt{\frac{2}{\pi}}\left\{0 - \frac{1}{a^2+w^2}\left(w\cdot sin(0) + (-a)cos(0)\right)\right\} = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+w^2}, w \in R
\end{aligned}$$

b) $F_c^{-1}\left\{\frac{1}{1+w^2}\right\} = f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{1+w^2}cos(wx)dx =$ By Laplace integral,

$$\sqrt{2\pi}\cdot\frac{\pi}{2}e^{-x} = \frac{\sqrt{\pi}}{\sqrt{2}}e^{-x} = \frac{\sqrt{2\pi}}{2}e^{-x}, x > 0$$

c)

$$\begin{aligned}
F_s\left\{e^{-ax}\right\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(w)sin(wx)dx = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}sin(wx)dx = \\
&= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+w^2}\left((-a)sin(wx) - w\cdot cos(wx)\right)\right]\Big|_0^\infty = \\
&= \sqrt{\frac{2}{\pi}}\left[0 - \frac{1}{a^2+w^2}\left\{(-a)sin(0) - w\cdot cos(0)\right\}\right] = \sqrt{\frac{2}{\pi}}\frac{w}{a^2+w^2}, w \in R
\end{aligned}$$

d)

$$\begin{aligned}
F_s^{-1}\left\{\frac{w}{a^2+w^2}\right\} &= f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(w)sin(wx)dw = \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{w\cdot sin(wx)}{a^2+w^2}dw = \sqrt{\frac{2}{\pi}}\left[\frac{\pi}{2}e^{-ax}\right] = \\
&= \frac{\sqrt{2\pi}}{2}e^{-ax}, x > 0
\end{aligned}$$

2. Calculate the following transforms.
a)$F\{f\}$ where $f(x) = \delta(x - x_0), x_0 \in R$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \delta(x - x_0)e^{-iwx}dx = \frac{1}{\sqrt{2\pi}}e^{-iwx_0}$$

b)$F\{f\}$ where $f(x) = e^{-k_0|x|}, k_0 \in R^+$

$$\begin{aligned}
f(x) &= \begin{cases} e^{-k_0 x} & x > 0 \\ e^{k_0 x} & x < 0 \end{cases} \\
\hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{k_0 x} e^{iwx} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-k_0 x} e^{iwx} dx = \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(k_0+iw)x}}{k_0 + iw} \bigg|_{-\infty}^{0} + \frac{e^{(iw-k_0)}}{iw - k_0} \bigg|_{0}^{\infty} \right] = \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{k_0 + iw} - \frac{1}{iw - k_0} \right] = \frac{1}{\sqrt{2\pi}} \left[\frac{-2k_0}{-w^2 - k_0^2} \right] = \\
&= \sqrt{\frac{2}{\pi}} \left(\frac{k_0}{w^2 + k_0^2} \right)
\end{aligned}$$

c) $F^{-1}\{\hat{f}\}$ where $\hat{f}(w) = \frac{1}{2}[\delta(w + w_0) + \delta(w - w_0), w_0 \in R$

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}} (\frac{1}{2}) \int_{-\infty}^{\infty} [\delta(w + w_0) + \delta(w - w_0)]\, e^{iwx} dw = \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{iw_0 x} + e^{-iw_0 x}}{2} \right] = \frac{1}{\sqrt{2\pi}} cos(w_0 x)
\end{aligned}$$

d) $F^{-1}\{\hat{f}\}$ where $\hat{f}(w) = \frac{1}{2}[\delta(w + w_0) - \delta(w - w_0), w_0 \in R$

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}} (\frac{1}{2}) \int_{-\infty}^{\infty} [\delta(w + w_0) - \delta(w - w_0)]\, e^{iwx} dw = \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{iw_0 x} - e^{-iw_0 x}}{2} \right] = \frac{i}{\sqrt{2\pi}} sin(w_0 x)
\end{aligned}$$

e)$F\left\{f(x+c)\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x+c) e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-iw(u-c)} du = e^{iwc} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-iwu} du = e^{iwc} \hat{f}(w)$

3. The convolution of 2 functions is defined as:

$$(f * g) = \int_{-\infty}^{\infty} f(p) g(x-p) dp = \int_{-\infty}^{\infty} f(x-p) g(p) dp \tag{1}$$

a) Show that $F\{f * g\} = \sqrt{2\pi} F\{f\} F\{g\}$

$$\begin{aligned}
F\{f * g\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(p) g(x-p) dp \right] e^{-iwx} dx = \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p) g(x-p) e^{-iwx} dx dp = \qquad \text{let } x - p = q \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow x = q + p \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p) g(q) e^{-iw(p+q)} dx dp =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(p)e^{-iwp}dp \cdot \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(q)e^{-iwq}dq = \\
&= \frac{1}{\sqrt{2\pi}}\left[\sqrt{2\pi}F\{f\}\sqrt{2\pi}F\{g\}\right] = \sqrt{2\pi}F\{f\}F\{g\}
\end{aligned}
$$

b) Find the convolution of $f(x) = \delta(x - x_0)$ and $g(x) = e^{-x}$

$$(f * g)(x) = \int_{-\infty}^{\infty} \delta(p - x_0)e^{-(x-p)}dp = -e^{-(x-x_0)}$$

old.a) Write down the integral definitionof auto-correlation and cross-correlation.

Cross-correlation: $(f \star g)(x) = \int f^{\star}(t)g(x + t)dt$
Auto-correlation: $\bar{f}(-\tau) \star f(\tau) = \int_{-\infty}^{\infty} f(t + \tau)\bar{f}(t)dt$

old.b) Compare and contrast cross-correlation and convolution.

The probability distribution of the difference -x+y is given by the cross-correlation while the convolution gives the probability distribution of the sum x+y.
(x,y are independent random variables)

old.c) Describe an application of cross-correlation related to signal processing.

Cross-correlation, in signal processing, is commonly used to find features in an unknown signal by comparing it to a known one.

4. Given the ODE

$$
\begin{aligned}
&y' + y = f(x) \quad -\infty < x < \infty \\
&\text{let } f(x) = \delta(x)
\end{aligned}
\tag{2}
$$

a) Calculate the frequency response

$$
\begin{aligned}
F\{y' + y\} &= F\{f(x)\} \\
&= iw\hat{y} + \hat{y} = \hat{f} \\
&= \hat{y}(1 + iw) = \hat{f} \\
\hat{y}(w) &= \hat{f}(w)\frac{1}{iw + 1} \leftarrow \text{Frequency response}
\end{aligned}
$$

b) Calculate the Green's Function
Let $\left(\frac{1}{iw+1}\right) = \hat{g}(w)$ and $F^{-1}\{\hat{g}\} = g(t) = \sqrt{2\pi}e^{-t}, t > 0$

$$
\begin{aligned}
F^{-1}\{\hat{y}(w)\} &= y(t) = F^{-1}\{\hat{f}(w)\hat{g}(w)\} \\
y(t) &= \frac{(f * g)(t)}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(p)g(t - p)dt \\
y(t) &= \int_{0}^{\infty} \delta(p)e^{-(t-p)}dp
\end{aligned}
$$

Green's Function: $e^{-(t-p)}$
c) Using convolution, find the steady-state solution.(2 pts extra)

$$y(t) = \int_0^{\infty} \delta(p) e^{-(t-p)} dp = e^{-t}$$