E. Kreyszig, Advanced Engineering Mathematics, 8^{th} ed.	Section 7.1, pgs. 272-278
Lecture: Matrices, Vectors: Algebra of '+'	Module: 02
Suggested Problem Set: Suggested Problems : {5,7}	January 8, 2009

E. Kreyszig, Advanced Engineering Mathematics, 8^{th} ed.	Section 7.2, pgs. 278-287
Lecture: Matrix Multiplication: Algebra of '·'	Module: 03
Suggested Problem Set: Suggested Problems : {3, 5, 8, 13, 19, 20, 22}	January 8, 2009

Quote of Lecture 2
Remember where the thought is. I brought all this so you could servive when law is lawless.
Gorillaz: Clint Eastwood (2001)

We begin the course with the algebra of matrices. To do this we must do the following:

- Define a mathematical object called a **matrix**.
- Define how groups of **matrices** behave with respect to the binary operations '+' and '·'.

once this is completed we can begin to study how this algebra relates to linear systems and how we should think about solutions to linear problems. You have studied linear equations before in the sense that you have asked the question, for what values of x does the equation,

$$ax = b, \quad a, b \in \mathbb{R}, \tag{1}$$

have a solution? It should be straightforward to see that for a = 0 the problem has the unique solution,

$$x = \frac{b}{a}, \quad a \neq 0. \tag{2}$$

What about the case where $a = 0$? Well, that is trickier. If $b = 0$ then the value of x doesnt matter. We always have equality, there are infinitely many choice for x! However, in the case where $b \neq 0$ we have the inconsistent equation, $0 \cdot x = b \neq 0$. There are no values of x, which will satisfy the equation.

However, we are getting ahead of ourselves. The question we need to ask now is given a lot of data how can we systematically organize it? Also, once this is done how can we manipulate groups of them? Without answers to these questions we have no hope of setting up equations (1)-(2) for large data. In what follows will record the definitions of **matrices** and their **algebraic structure**.[1]

Goals

- Understand the vocabulary describing the objects and operations of linear algebra.
- Know the algebraic rules of matrix addition, scaling, multiplication, conjugation and transposition and how these rules correspond to linear transformations/systems.

Objectives

- Associate vocabulary with objects and their notations.
- Define and practice the allowed algebraic operations for matrices highlighting how this relates to to linear systems/transformations
- Record the rules of this algebra highlighting key differences with other common algebraic structures.

[1]From the perspective of abstract algebra, a field that studies collections of objects and their properties with respect to binary operators, we would call this sort of structure a non-commutative ring.

Definition: Matrix - A *matrix* is a set of objects organized by two indices into a rectangular array. In the case that these objects exist in the set of complex numbers we write $\mathbf{A} \in \mathbb{C}^{m\times n}$, where $n, m \in \mathbb{N}$. At the element level we have that:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}, \text{ where } [\mathbf{A}]_{ij} = a_{ij},\ a_{ij} \in \mathbb{C}, \text{ for } i = 1, 2, 3, \cdots, m \text{ and } j = 1, 2, 3, \cdots, n. \quad (3)$$

- In the case that $n = m$ we call the matrix **square**. Otherwise it is called **rectangular**.
- For a square matrix the entries running from the upper left to the lower right are called the **main diagonal** entries.

Definition: Vector - A *column vector*, or just vector, is matrix of size, $m \times 1$ where $m \in \mathbb{N}$. A *row vector* is matrix of size, $1 \times n$ where $n \in \mathbb{N}$ and if $\mathbf{v} \in \mathbb{C}^{m\times 1}$ or $\mathbf{r} \in \mathbb{C}^{1\times n}$ and at the element level we have that:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix}, \text{ where } v_i \in \mathbb{C} \text{ for } i = 1, 2, 3, \cdots m. \quad (4)$$

$$\mathbf{r} = \begin{bmatrix} r_1 & r_2 & r_3 & \dots & r_n \end{bmatrix}, \text{ where } r_j \in \mathbb{C}, \text{ for } j = 1, 2, 3, \cdots n. \quad (5)$$

Definition: Scalar - A *scalar* is a matrix whose size is 1×1. That is, a *scalar* is an element of the complex number system, $s \in \mathbb{C}$.

Defintion: Equality of Matrices - Two matrices $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{m\times n}$ are said to be equal if and only if $a_{ij} = b_{ij}$ for $i = 1, 2, 3, \dots, m$ and $j = 1, 2, 3, \dots, n$.

Defintion: Addition and Scalar Multiplication of Matrices - Let $\mathbf{A} \in \mathbb{C}^{m\times n}$and $\mathbf{B} \in \mathbb{C}^{m\times n}$ then $\mathbf{A} + \mathbf{B} = \mathbf{C}$ is defined such that $\mathbf{C} \in \mathbb{C}^{m\times n}$ where $c_{ij} = a_{ij} + b_{ij}$ for $i = 1, 2, 3, \dots, m$ and $j = 1, 2, 3, \dots, n$. Also, let $s \in \mathbb{C}$, then $s\mathbf{A} = \mathbf{C}$, where $c_{ij} = s \cdot a_{ij}$ for $i = 1, 2, 3, \dots, m$ and $j = 1, 2, 3, \dots, n$. Following from these definitions we have the rules for matrix addition and scalar multiplication as:

1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
2. $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
3. $\mathbf{A} + \mathbf{0} = \mathbf{A}$
4. $\mathbf{A} + -1 \cdot \mathbf{A} = \mathbf{0}$
5. $r(\mathbf{A} + \mathbf{B}) = r\mathbf{A} + r\mathbf{B}$
6. $(r + s)\mathbf{A} = r\mathbf{A} + s\mathbf{A}$
7. $r(s\mathbf{A}) = (rs)\mathbf{A}$
8. $1 \cdot \mathbf{A} = \mathbf{A}$

where $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C} \in \mathbb{C}^{m\times n}$ and $r, s \in \mathbb{C}$

Defintion: Transposition - Given$\mathbf{A} \in \mathbb{R}^{m\times n}$, define the transpose of $\mathbf{A}$ to be the matrix $\mathbf{A}^{\mathrm{T}} \in \mathbb{R}^{n\times m}$, such that:

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \dots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \dots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \dots & a_{m3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \dots & a_{nm} \end{bmatrix} \tag{6}$$

- If $\mathbf{A}$ is such that $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$ then the matrix $\mathbf{A}$ is called symmetric.
- If $\mathbf{A}$ is such that $-\mathbf{A} = \mathbf{A}^{\mathrm{T}}$ then the matrix $\mathbf{A}$ is called skew-symmetric.

Defintion: Conjugation - Given $\mathbf{A} \in \mathbb{C}^{m\times n}$, define the conjugate of $\mathbf{A}$ to be the matrix $\bar{\mathbf{A}} \in \mathbb{C}^{n\times m}$ such that,

$$\bar{\mathbf{A}} = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} & \bar{a}_{13} & \dots & \bar{a}_{1n} \\ \bar{a}_{21} & \bar{a}_{22} & \bar{a}_{23} & \dots & \bar{a}_{2n} \\ \bar{a}_{31} & \bar{a}_{32} & \bar{a}_{33} & \dots & \bar{a}_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{m1} & \bar{a}_{m2} & \bar{a}_{m3} & \dots & \bar{a}_{mn} \end{bmatrix}. \tag{7}$$

- The bar implies complex conjugation. That is if $c \in \mathbb{C}$ then $c = a + bi,\ a, b \in \mathbb{R}$ and $\bar{c} = a - bi$.

Defintion: Adjoint - Given $\mathbf{A} \in \mathbb{C}^{m\times n}$, define the adjoint of $\mathbf{A}$ to be the matrix $\mathbf{A}^{\mathrm{H}} \in \mathbb{C}^{n\times m}$ such that $\mathbf{A}^{\mathrm{H}} = (\bar{\mathbf{A}})^{\mathrm{T}} = \overline{(\mathbf{A}^{\mathrm{T}})}$.

- The adjoint is considered as an extension of the transpose to matrices with complex numbers. Sometimes the adjoint is called the Hermitian of a matrix.
- A matrix is called self-adjoint or Hermitian if $\mathbf{A}^{\mathrm{H}} = \mathbf{A}$.

Defintion: Matrix Multiplication - Let $\mathbf{A} \in \mathbb{C}^{m\times n}$and $\mathbf{B}\in \mathbb{C}^{p\times q}$. If $n = p$ then $\mathbf{AB}$=$\mathbf{C}$ is defined such that $\mathbf{C}\in \mathbb{C}^{m\times q}$ where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. The general properties for matrix products are:

1. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
2. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
3. $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$
4. $r(\mathbf{AB}) = r(\mathbf{A})\mathbf{B} = \mathbf{A}r\mathbf{B}$
5. $\mathbf{I}_n\mathbf{A} = \mathbf{A} = \mathbf{AI}_m$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are defined appropriately and $r \in \mathbb{C}$

- It is not necessarily the case that $\mathbf{AB}$=$\mathbf{BA}$. That is, matrix multiplication does not, in general, commute.
- The identity matrix $\mathbf{I}_k$ is a square matrix such that $[\mathbf{I}_{k\times k}]_{ij} = 1$ if $i = j$ and $[\mathbf{I}_{k\times k}]_{ij} = 0$ if $i \neq j$.
- The inverse matrix of a square matrix $\mathbf{A}$ is the square matrix $\mathbf{A}^{-1}$ such that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Defintion: Inner Product - Given $\mathbf{x}\in \mathbb{C}^{n\times 1}$ and $\mathbf{y}\in \mathbb{C}^{n\times 1}$ define the inner product of $\mathbf{x}$ and $\mathbf{y}$ to be:

$$\mathbf{x}\cdot\mathbf{y} = \mathbf{x}^{\mathrm{H}}\mathbf{y} = \sum_{i=1}^{n} x_i y_i \tag{8}$$

- Using the inner product it is possible to define matrix multiplication as $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = \mathbf{a}_i \cdot \mathbf{b}_j$ where $\mathbf{a}_i$ is the ith row of $\mathbf{A}$ and $\mathbf{b}_j$ is the jth column of $\mathbf{B}$.