Note Title 11/15/2006

11/15/06

$$\nabla^2 \psi(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right)$$

$$+ \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right)$$

$$+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} = 0$$

Laplaces equation in spherical coordinates

Make our normal Sep. of variables guess

$$\psi(r,\theta,\phi) = R(r)\,P(\theta)\,Q(\phi)$$

So, e.g. $\frac{\partial \psi}{\partial r} = R'(r)\,P(\theta)\,Q(\phi)$

$$\nabla^2\psi = \frac{1}{r^2} P\Phi \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right)$$

$$+ \frac{R\Phi}{r^2 \sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right)$$

$$\frac{RP}{r^2 \sin^2\theta} \frac{d^2\Phi}{d\varphi^2} = 0$$

Always now divide by $RP\Phi$

$$\frac{\nabla^2\psi}{\psi} = \frac{1}{r^2 R} \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right)$$

$$+ \frac{1}{r^2 \sin\theta \, P} \frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right)$$

$$+ \frac{1}{r^2 \sin^2\theta \, \Phi} \frac{d^2\Phi}{d\varphi^2} = 0$$

we can isolate the $\Phi(\phi)$ part if we multiply by $r^2 \sin^2\theta$, $\nabla^2 \psi = 0 \Rightarrow$

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right)$$

$$+ \frac{\sin\theta}{P}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right)$$

$$= -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} \equiv m^2$$

$$\boxed{\Phi'' + m^2 \Phi = 0}$$ first ODE

so $\Phi(\phi) \propto e^{im\phi}$

Does m have to be an integer?

$Q(\varphi) \propto e^{im\varphi}$

Suppose $m = .1$

$Q(0) = 1$

$Q(2\pi) = e^{i \cdot .1 \cdot 2\pi} \neq 1$

Like a Boundary condition the periodicity forces m to be an integer

Back to r, θ:

$$\frac{\sin^2\theta}{R} \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right)$$

$$+ \frac{\sin\theta}{P} \frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) = m^2$$

Divide by $\sin^2\theta$ and move θ part to RHS

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)$$

$$= -\frac{1}{\sin\theta}\frac{1}{P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{m^2}{\sin^2\theta}$$

r on the left $\qquad$ θ on right

r is easiest to do first

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = k^2$$

Guess a trial Solution

$$R(r) = A r^{\alpha}$$

$$R' = A\alpha r^{\alpha-1}$$

$$r^2 R' = A\alpha r^{\alpha+1}$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) =$$

$$\frac{1}{\cancel{A}\,\cancel{r^\alpha}} \cdot \cancel{A}\,(\alpha+1)\alpha\,\cancel{r^\alpha} = k^2$$

in order that $R = A r^\alpha$ be a sol'n.

$$\Rightarrow \quad \alpha(\alpha+1) = k^2$$

So far we don't know about k^2 but suppose $k^2 = \ell(\ell+1)$

$$\alpha(\alpha+1) = \ell(\ell+1)$$

$$\Leftrightarrow \quad (\alpha-\ell)(\alpha+(\ell+1)) = 0$$

So either $\boxed{\alpha = \ell \quad \text{or} \quad \alpha = -(\ell+1)}$

recap: we can get solutions to the radial part of Laplace's eqn if we write the separation constant as

$$k^2 = \ell(\ell+1)$$

Then $R(r) = A r^\alpha$ works for $\alpha = \ell$, $\alpha = -(\ell+1)$

so

$$\boxed{R(r) = A_\ell r^\ell + B_\ell r^{-(\ell+1)}}$$

second ODE solution

Finally we get to the
Θ part.

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]P=0$$

Legendres equation

Exercise Let $x=\cos\theta$

Show that (above) becomes

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]P(x)$$

Solutions of this equation
involve 2 indices ℓ, m.

Lets call them

$$P_{\ell,m}(x)$$

Then $\psi(r,\theta,\varphi) = R(r)P(\theta)\Phi(\varphi)$

$$= \begin{cases} r^{\ell} \; P_{\ell m}(\cos\theta)\, e^{im\varphi} \\ r^{-(\ell+1)} \; P_{\ell m}(\cos\theta)\, e^{im\varphi} \end{cases}$$

$$\underbrace{P_{\ell m}(\cos\theta)}_{\text{Legendre polynomials}} e^{im\varphi} \equiv \underbrace{Y_{\ell m}(\theta,\varphi)}_{\text{spherical Harmonics}}$$

Major Result!

Any solution of $\nabla^2 \psi = 0$ can be written as

$$\boxed{\psi(r,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(A_{lm} r^{l} + B_{lm} r^{-(l+1)} \right) Y_{lm}}$$

we have not yet proved that m runs from $-l, l$ but enough for today.

Its not that bad, really.

$$Y_{00} = \sqrt{\frac{1}{4\pi}}$$

$$Y_{10} = \sqrt{3/4\pi} \cos\theta$$

$$Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{\pm i\varphi}$$

So any spherically symmetric solution of $\nabla^2\psi = 0$ can only involve $l=0, m=0$

$$\psi(r,\theta,\varphi) = \left(A_{00} r^0 + B_{00} r^{-1}\right) Y_{00}$$

$\psi = \frac{1}{r}$ since $Y_{00} = \sqrt{\frac{1}{4\pi}}$

$B_{00} = \sqrt{4\pi}$