MATH332 - April 11, 2010
Exam II - 50 Points

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In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (10 Points)

   (a) True/False : No Justification Required

      i. Any collection of $n+1$ many vectors from $\mathbb{R}^n$ forms a linearly dependent set.

         True

      ii. If $\mathbf{A} \in \mathbb{R}^{m \times n}$ is onto then Rank $\mathbf{A} = m$.

         True

      iii. The column space of $\mathbf{A}$ is the set of all solutions to $\mathbf{Ax} = \mathbf{b}$.

         False

      iv. If an $n \times n$ matrix has n-many linearly independent eigenvectors then it is invertible.

         False [Well it might be true sometimes but not always

      v. The set of all $n-$degree polynomials, $p(t)$, such that $p(2) = 3$ forms a vector subspace of $\mathbb{P}_n$.

         False,

   (b) Short Response

      i. What must be true of $\mathbf{A}_{n \times n}$ for it to admit a diagonalization? Also, briefly explain why a diagonaization is important/useful.

         If $A_{n\times n}$ is s.t. $A = PDP^{-1}$ then A must have n-many linearly independent Eigenvectors. When this is true $A\vec{x} = \vec{b} \Leftrightarrow D\vec{y} = \vec{c}$ where D is diagonal (totally decoup.) and $\vec{y} = P^{-1}\vec{x}$, $\vec{c} = P^{-1}\vec{b}$.

      ii. Given a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, what are the definitions for its null and column spaces? Why are these spaces important/useful.

         $\text{Col}\,A = \{\vec{b} \in \mathbb{R}^m : \vec{b} = A\vec{x}\}$ ≡ Defines where the objects can be placed so they intersect.

         $\text{Nul}\,A = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$ ≡ Defines how the linear objects intersect

2. (10 Points) Quickies

(a) Determine the dimension of the vector space formed by the span of each of the three sets.

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\}, \quad S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad S_3 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$$

dim span $S_1 = 2$ dim Span $S_2 = 2$ dim Span $S_3 = 2$

(b) Given,

$$\mathbf{A}_1 = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 8 & 6 & 4 & 2 \\ 12 & 9 & 6 & 3 \\ 16 & 12 & 8 & 4 \end{bmatrix}. \quad (1)$$

Find one eigenvalue of $\mathbf{A}_1$ and one eigenvalue of $\mathbf{A}_2$.

$A_1\bar{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \lambda = 1$ is an Eigenvalue of A_1

$\det(A_2) = 0 \Rightarrow \lambda = 0$ is an Eigenvalue of A_2

(c) Given,

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{x}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_5 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

i. Which of the previous vectors are eigenvectors of $\mathbf{A}$ and what are their corresponding eigenvalues?

$A_1\vec{x}_1 = \vec{0}, \quad \lambda = 0$

$A_1\vec{x}_2 = \vec{0}, \quad \lambda = 0$

$A_1\vec{x}_3 = \vec{0}$ ⟵ trivial vector is not an E-vector

$A_1\vec{x}_4 = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 2\vec{x}_4, \quad \lambda = 2$

$A_1\vec{x}_5 = 2\vec{x}_5, \quad \lambda = 2$

ii. Which of the previous vectors are in the null-space of $\mathbf{A}$?

x_1, x_2, x_3

iii. Which of the previous vectors are in the column-space of $\mathbf{A}$?

x_3, x_4, x_5

3. (10 Points) Theory

has dim 1×1

(a) Let $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{A} \in \mathbb{R}^{n\times n}$ such that $\mathbf{A} = \mathbf{A}^\mathrm{T}$. Prove that $\bar{\mathbf{x}}^\mathrm{T}\mathbf{A}\mathbf{x}$ is a real number.

$$\bar{q} = \overline{(\bar{x}^T A x)} = x^T \bar{A} \bar{x} = (\bar{x}^T \bar{A}^T x)^T = (\bar{x}^T A x)^T = q^T = q$$

$\bar{x}^T A x = q$

$\Rightarrow q \in \mathbb{R}$

(b) Prove that the null space of a matrix $\mathbf{A}_{m\times n}$ is a vector subspace of $\mathbf{R}^n$.

Let $x_1, x_2 \in Nul(A) \subset \mathbb{R}^n$ by Prob 1

$A(c_1x_1 + c_2x_2) = c_1Ax_1 + c_2Ax_2 = 0 + 0 = 0 \Rightarrow c_1x_1 + c_2x_2 \in Nul\,A$

if $c_1 = c_2 = 0$ the proof still holds $\Rightarrow \bar{0} \in Nul A \Rightarrow Nul A$ is a subspace

(c) Suppose that $\mathbf{A} \in \mathbb{R}^{m\times n}$ such that $m < n$ provide an upper and lower bound for the dimension of the null space of $\mathbf{A}$.

$\Rightarrow$ free var $\Rightarrow$ dim Nul A > 0

$\Rightarrow \quad 0 < \dim Nul A \leq m$

→ Equality only if $A = 0$

4. (10 Points) Given,

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}.$$

Determine a basis and the dimension of the null-space and column-space of $\mathbf{A}$.

$$A \sim \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 9 & -2 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

✓ ✓ ✓

$\Rightarrow x_4 = 3x_5$

$x_3 = \frac{x_4 - x_5}{3} = -\frac{4}{3}x_5$

$x_1 = \frac{1}{2}(3x_2 - 6x_3 - 2x_4 - 5x_5) =$

$= \frac{1}{2}(3x_2 - 6(-\frac{4}{3}x_5) + 6x_5 - 5x_5) =$

$= \frac{1}{2}(3x_2 + 9x_5) = \frac{3}{2}x_2 + \frac{9}{2}x_5$

$$\Rightarrow B_{Nul(A)} = \left\{ \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix} \right\} \quad \dim(Nul(A)) = 2$$

$$B_{col(A)} = \left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix} \right\}$$

$\dim(col(A)) = 3$

5. (10 Points) Given,

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}. \quad , (A - \lambda I) = \begin{bmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 6 \\ 2 & 0 & -\lambda \end{bmatrix}$$

(a) Find the eigenvalues of **A**.

$$\det(A - \lambda I) = -\lambda(2-\lambda)(-\lambda) + 2(-2(2-\lambda)) =$$

$$= (\lambda^2 + -4)(2-\lambda) = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 2, \lambda_3 = -2$$

(b) Find the eigenvectors of **A**.

Case $\lambda_1 = 2$:

$$[A - \lambda_1 I \mid 0] = \left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = x_3 \\ x_2, x_3 \in \mathbb{R} \end{array} \quad \vec{x}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{x}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Case $\lambda_3 = -2$:

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 4 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] \Rightarrow x_2 = 0 \quad x_1 = -x_3 \Rightarrow \vec{x}^{(3)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$[P \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & +1/2 & 0 & -1/2 \end{array}\right]$$

(c) Using these results calculate $\mathbf{A}^8$.

$$A^8 = PD^8P^{-1}, \text{ where } D^8 = \begin{bmatrix} 2^8 & 0 & 0 \\ 0 & 2^8 & 0 \\ 0 & 0 & (-2)^8 \end{bmatrix} = 256 \cdot I$$

$$\Rightarrow A^8 = P \cdot 256 \cdot I \cdot P^{-1} = 256 \cdot I.$$

6. Extra Credit Questions:

(a) Define the following set,

$$\Pi_p = \{f : \mathbb{R} \to \mathbb{R} : f(x+p) = f(x),\, p \in \mathbb{R}^+\}. \tag{2}$$

Show that Π_p forms a subspace of the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$.

Let $c_1, c_2 \in \mathbb{R}$, $f_1, f_2 \in \Pi_p$

Then for $f = c_1 f_1 + c_2 f_2$

$f(x+p) = c_1 f_1(x+p) + c_2 f_2(x+p) = c_1 f_1(x) + c_2 f_2(x) = f(x)$

$\Rightarrow f \in \Pi_p$

If $c_1 = c_2 = 0$ then $f \equiv 0$ and $f(x+p) = f(x)$.

Thus, Π_p forms a subspace.

(b) Assume that $\mathbf{A} \in \mathbb{R}^{n \times n}$ is diagonalizable and has $\lambda = 1$ with algebraic multiplicity of n. Prove that $\mathbf{A}$ is such that $\mathbf{A} = \mathbf{A}^T$ and that $\mathbf{A}^{-1} = \mathbf{A}^T$.

If $\sigma(A) = \{\underbrace{1, \cdots, 1}_{n\text{-times}}\}$ then $A = PDP^{-1} = PIP^{-1} = I$

We note

$A = I \Rightarrow$

$I^T = I$ and $I \cdot I = I \Rightarrow I^{-1} = I = I^T$

This is the only matrix for which this is true.

(c) Prove that if $\mathbf{A} = \mathbf{PBP}^{-1}$, where all the matrices are square, then $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues.

$\det(A - \lambda I) = \det(PBP^{-1} - PP^{-1}) =$

$= \det(P(B - \lambda I)P^{-1}) =$

$= \det(D)\det(B - \lambda I)\det(P^{-1}) =$

$= \det(P)\det(B - \lambda I)\frac{1}{\det(P)} = \det(B - \lambda I)$