ential equation, for now let us first discuss a simpler problem. If the initial traffic density is a constant,

$$\rho(x, 0) = \rho_0,$$

independent of x, then the density should remain constant (since all cars must move at the same speed). This is verified by noting that a constant density,

$$\rho(x, t) = \rho_0,$$

satisfies the partial differential equation 66.1. Any constant density is an **equilibrium*** density. Let us imagine driving in a traffic situation in which the density is approximately constant. What does your experience tell you? What kinds of phenomena do you observe? Does the density seem to stay constant? Some of you have probably had the experience of driving at a steady speed and all of a sudden, for no apparent reason, the car in front slows down. You must slow down, then the car behind slows down, and so on. Let us investigate that situation, namely one in which the density is *nearly* constant.

If the density is nearly uniform, then there should be an approximate solution to the partial differential equation such that

$$\boxed{\rho(x, t) = \rho_0 + \epsilon\rho_1(x, t),} \tag{66.2}$$

where $|\epsilon\rho_1| \ll \rho_0$. $\epsilon\rho_1(x, t)$ is called the perturbed traffic density (or the displacement from the constant density ρ_0). Assume that the initial density is a known function of x, nearly equaling the constant ρ_0,

$$\rho(x, 0) = \rho_0 + \epsilon f(x).$$

Thus the perturbed traffic density is also known initially, $\rho_1(x, 0) = f(x)$. Substituting equation 66.2 into the second form of the partial differential equation 66.1b yields

$$\epsilon\frac{\partial \rho_1}{\partial t} + \frac{dq}{d\rho}(\rho_0 + \epsilon\rho_1(x, t))\epsilon\frac{\partial \rho_1}{\partial x} = 0,$$

where a power of ϵ has been cancelled. The derivative $dq/d\rho$ is evaluated at the total traffic density $\rho_0 + \epsilon\rho_1(x, t)$. Expanding that expression via a Taylor series, yields

$$\frac{dq}{d\rho}(\rho_0 + \epsilon\rho_1(x, t)) = \frac{dq}{d\rho}(\rho_0) + \epsilon\rho_1\frac{d^2q}{d\rho^2}(\rho_0) + \frac{(\epsilon\rho_1)^2}{2!}\frac{d^3q}{d\rho^3}(\rho_0) + \cdots$$

Thus to leading order (that is neglecting the small terms) the following equation is obtained:

*An equilibrium solution of a partial differential equation is a *solution* which does not depend on time.

$$\boxed{\frac{\partial \rho_1}{\partial t} + \frac{dq}{d\rho}(\rho_0)\frac{\partial \rho_1}{\partial x} = 0.} \tag{66.3}$$

This partial differential equation governs the perturbed traffic density. However, equation 66.3 is a *linear* partial differential equation while the exact traffic equation 66.1 is a *nonlinear* partial differential equation. The coefficient that appears in equation 66.3, $(dq/d\rho)(\rho_0)$, is a constant (the slope of the flow as a function of density evaluated at the constant density). This resulting partial differential equation is the simplest kind involving both partial derivatives,

$$\boxed{\frac{\partial \rho_1}{\partial t} + c\frac{\partial \rho_1}{\partial x} = 0,} \tag{66.4}$$

where $c = (dq/d\rho)(\rho_0)$.

EXERCISES

66.1. Assume that $dq/d\rho = a + b\rho$

(a) What nonlinear partial differential equation describes conservation of cars?

(b) Describe all possible (time-independent) traffic densities $\rho(x)$ which satisfy the equation of part (a).

(c) Show that $\rho = \rho_0$ (any constant) is a time-independent solution.

(d) By substituting

$$\rho(x, t) = \rho_0 + \epsilon\rho_1(x, t)$$

into the partial differential equation of part (a) and by neglecting nonlinear terms, what equation does $\rho_1(x, t)$ satisfy? Is it the same as equation 66.3?

67. A Linear Partial Differential Equation

In this section, we will solve the partial differential equation corresponding to the linearization of the traffic flow problem,

$$\boxed{\frac{\partial \rho_1}{\partial t} + c\frac{\partial \rho_1}{\partial x} = 0,} \tag{67.1}$$