Reading:

Today: G 8.2

Monday: G 9.1

Conservation Laws:

Charge

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t}$$

Energy

$$\vec{\nabla}\cdot\vec{S} = -\frac{\partial u_{em}}{\partial t} - \vec{J}\cdot\vec{E} = -\frac{\partial(u_{tot})}{\partial t}$$

What about conservation of momentum

$$\vec{F} = \frac{d\vec{p}}{dt} = q\vec{E} + q\vec{v}\times\vec{B} = \int_V \rho\vec{E} + \rho\vec{v}\times\vec{B}\, dV$$

$$\frac{d\vec{F}}{dV} = \vec{f} = \text{force per unit volume}$$
$$= \rho\vec{E} + \vec{J}\times\vec{B}$$

Let's get rid of ρ and $\vec{J}$.

Maxwell's eqn ① $\rho = \epsilon_0 \vec{\nabla}\cdot\vec{E}$

④ $\vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}$

$$\rho\vec{E} + \vec{J}\times\vec{B} = \left[\epsilon_0\vec{\nabla}\cdot\vec{E}\right]\vec{E} + \left[\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}\right]\times\vec{B}$$

$$\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \frac{\partial\vec{E}}{\partial t}\times\vec{B} + \vec{E}\times\frac{\partial\vec{B}}{\partial t}$$

$$\frac{\partial\vec{E}}{\partial t}\times\vec{B} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \vec{E}\times\frac{\partial\vec{B}}{\partial t}$$

Maxwell ② $\frac{\partial\vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}$

$$\frac{\partial\vec{E}}{\partial t}\times\vec{B} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times(\vec{\nabla}\times\vec{E})$$

$$\begin{aligned}\rho\vec{E} + \vec{J}\times\vec{B} = {} & \epsilon_0\vec{E}(\vec{\nabla}\cdot\vec{E}) + \frac{1}{\mu_0}\vec{B}(\vec{\nabla}\cdot\vec{B}) \\ & - \frac{1}{\mu_0}\vec{B}\times(\vec{\nabla}\times\vec{B}) - \epsilon_0\vec{E}\times(\vec{\nabla}\times\vec{E}) \\ & - \frac{\partial}{\partial t}(\epsilon_0\vec{E}\times\vec{B})\end{aligned}$$

$$\vec{E}\times(\vec{\nabla}\times\vec{E}) = \frac{1}{2}\vec{\nabla}(E^2) - (\vec{E}\cdot\vec{\nabla})\vec{E}$$

$$\left.\begin{aligned}\rho\vec{E} + \vec{J}\times\vec{B} = {} & \epsilon_0\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) + (\vec{E}\cdot\vec{\nabla})\vec{E}\right] \\ & + \frac{1}{\mu_0}\left[\vec{B}(\vec{\nabla}\cdot\vec{B}) + (\vec{B}\cdot\vec{\nabla})\vec{B}\right] \\ & - \frac{1}{2}\vec{\nabla}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\end{aligned}\right\} \vec{\nabla}\cdot\overleftrightarrow{T}$$

$$- \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) \longleftarrow \frac{d\vec{p}_{em}}{\partial t}$$

Enter the Maxwell Stress Tensor.

$$T_{ij} = \epsilon_0\left[E_iE_j - \frac{1}{2}\delta_{ij}E^2\right] + \frac{1}{\mu_0}\left[B_iB_j - \frac{1}{2}\delta_{ij}B^2\right]$$

$$\begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} \qquad \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

$$E_x^2 - \frac{1}{2}E_x^2 - \frac{1}{2}E_y^2 - \frac{1}{2}E_z^2$$

$$\begin{bmatrix} \epsilon_0\left(\frac{1}{2}E_x^2 - \frac{1}{2}E_y^2 - \frac{1}{2}E_z^2\right) + \frac{1}{\mu_0}\left(\frac{1}{2}B_x^2 - \frac{1}{2}B_y^2 - \frac{1}{2}B_z^2\right) & \epsilon_0 E_xE_y + \frac{1}{\mu_0}B_xB_y & \epsilon_0 E_xE_z + \frac{1}{\mu_0}B_xB_z \\ \epsilon_0 E_xE_y + \frac{1}{\mu_0}B_xB_y & \epsilon_0\left(-\frac{1}{2}E_x^2 + \frac{1}{2}E_y^2 - \frac{1}{2}E_z^2\right) + \frac{1}{\mu_0}\left(-\frac{1}{2}B_x^2 + \frac{1}{2}B_y^2 - \frac{1}{2}B_z^2\right) & \epsilon_0 E_yE_z + \frac{1}{\mu_0}B_yB_z \\ \epsilon_0 E_xE_z + \frac{1}{\mu_0}B_xB_z & \epsilon_0 E_yE_z + \frac{1}{\mu_0}B_yB_z & \epsilon_0\left(-\frac{1}{2}E_x^2 - \frac{1}{2}E_y^2 + \frac{1}{2}E_z^2\right) + \frac{1}{\mu_0}\left(-\frac{1}{2}B_x^2 - \frac{1}{2}B_y^2 + \frac{1}{2}B_z^2\right) \end{bmatrix}$$

$\vec{\nabla}$ operator in matrix form looks like

$$\left[\nabla_x \quad \nabla_y \quad \nabla_z\right] = \left[\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z}\right]$$

Let's find $(\vec{\nabla}\cdot\overleftrightarrow{T})_x$ just for $\vec{E}$ parts and $\vec{B}$ is identical.

$$\frac{\partial}{\partial x}\left[\frac{\epsilon_0}{2}(E_x^2 - E_y^2 - E_z^2)\right] + \frac{\partial}{\partial y}\left[\epsilon_0 E_xE_y\right] + \frac{\partial}{\partial z}\left[\epsilon_0 E_xE_z\right] \qquad \frac{\partial}{\partial x}(y^2) = 2y\frac{\partial y}{\partial x}$$

$$= \epsilon_0\left[E_x\frac{\partial E_x}{\partial x} - E_y\frac{\partial E_y}{\partial x} - E_z\frac{\partial E_z}{\partial x} + E_x\frac{\partial E_y}{\partial y} + E_y\frac{\partial E_x}{\partial y} + E_x\frac{\partial E_z}{\partial z} + E_z\frac{\partial E_x}{\partial z}\right]$$

$$= \epsilon_0\left[E_x\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) + \left(E_x\frac{\partial}{\partial x} + E_y\frac{\partial}{\partial y} + E_z\frac{\partial}{\partial z}\right)E_x - \underbrace{E_x\frac{\partial E_x}{\partial x}}_{\frac{1}{2}\frac{\partial}{\partial x}(E_x^2)} - \underbrace{E_y\frac{\partial E_y}{\partial x}}_{\frac{1}{2}\frac{\partial}{\partial x}(E_y^2)} - \underbrace{E_z\frac{\partial E_z}{\partial x}}_{\frac{1}{2}\frac{\partial}{\partial x}(E_z^2)}\right]$$

cancel out

$$= -\frac{1}{2}\frac{\partial}{\partial x}(E^2)$$

$$= \epsilon_0\left[E_x(\vec{\nabla}\cdot\vec{E}) + (\vec{E}\cdot\vec{\nabla})E_x - \frac{1}{2}\nabla_x(E^2)\right]$$

So $\vec{\nabla}\cdot\overleftrightarrow{T}$ does equal what we wanted.

Enough crappy math.

$$\rho\vec{E} + \vec{J}\times\vec{B} = \vec{\nabla}\cdot\overleftrightarrow{T} - \frac{\partial}{\partial t}\left[\epsilon_0\vec{E}\times\vec{B}\right] \qquad \frac{d\vec{p}_{mech}}{dt}$$

$$-\vec{\nabla}\cdot\overleftrightarrow{T} = -\frac{\partial}{\partial t}\left[\epsilon_0\vec{E}\times\vec{B}\right] - \left(\rho\vec{E} + \vec{J}\times\vec{B}\right)$$

$$\vec{P}_{em\,dens} = \epsilon_0\vec{E}\times\vec{B} = \epsilon_0\mu_0\vec{S} \qquad \left\{\frac{1}{\epsilon_0\mu_0} = c^2\right\}$$

$-\overleftrightarrow{T}$ is flow of momentum.

Last thing: angular momentum.

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\Downarrow$$

$$\boxed{\vec{\ell}_{em} = \vec{r} \times \underbrace{\vec{p}_{em,dens}}_{\epsilon_0 \vec{E} \times \vec{B}} = \epsilon_0 \left(\vec{r} \times (\vec{E} \times \vec{B})\right)}$$

Let's derive Maxwell's eqns in Gaussian units!

Define $\vec{e} = \sqrt{\epsilon_0}\,\vec{E}$

$\vec{b} = \frac{1}{\sqrt{\mu_0}}\vec{B}$

use the fact that $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$

$\rho' = \rho/(4\pi\sqrt{\epsilon_0})$ $\vec{J}' = \vec{J}/(4\pi\sqrt{\epsilon_0})$

find Maxwells eqn isn Gaussian units!

① $\frac{\rho}{\epsilon_0} = \vec{\nabla}\cdot\vec{E} = \frac{\rho' 4\pi\cancel{\sqrt{\epsilon_0}}}{\cancel{\epsilon_0}} = \frac{1}{\cancel{\sqrt{\epsilon_0}}}\vec{\nabla}\cdot\vec{e}$

① $\boxed{\vec{\nabla}\cdot\vec{e} = 4\pi\rho'}$

② $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = \frac{1}{\sqrt{\epsilon_0}}\vec{\nabla}\times\vec{e} = -\sqrt{\mu_0}\frac{\partial\vec{b}}{\partial t}$

$\Rightarrow \boxed{\vec{\nabla}\times\vec{e} = -\frac{1}{c}\frac{\partial\vec{b}}{\partial t}}$

③ $\vec{\nabla}\cdot\vec{B} = \emptyset \Rightarrow \boxed{\vec{\nabla}\cdot\vec{b} = 0}$

④ $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$

$\sqrt{\mu_0}\,\vec{\nabla}\times\vec{b} = 4\pi\sqrt{\epsilon_0}\,\mu_0\vec{J}' + \mu_0\sqrt{\epsilon_0}\frac{\partial\vec{e}}{\partial t}$

$\Rightarrow \vec{\nabla}\times\vec{b} = 4\pi\sqrt{\epsilon_0\mu_0}\,\vec{J}' + \sqrt{\epsilon_0\mu_0}\frac{\partial\vec{e}}{\partial t}$

$\boxed{\vec{\nabla}\times\vec{b} = \frac{4\pi}{c}\vec{J}' + \frac{1}{c}\frac{\partial\vec{e}}{\partial t}}$

What about $\vec{e}$ from a point charge.

$\vec{\nabla}\cdot\vec{e} = 4\pi\rho'$

$\oint\vec{e}\cdot d\vec{a} = 4\pi\int\rho' dv = 4\pi q'_{enc}$

$= e_r 4\pi r^2 \Rightarrow e_r = \frac{q'_{enc}}{r^2}$

$\boxed{\vec{e} = \frac{q'_{enc}}{r^2}\hat{r}}$

$\vec{F} = q\vec{E} + q\frac{\vec{v}}{c}\times\vec{B}$ in Gaussian units.

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