

To get full credit, you must show all of your work.

1. Using the definition of a Laplace <sup>transform</sup>, find the Laplace transform of  $f(t) = te^{4t}, s > 4$ .

Definition:  $\mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-st} dt$

$$\mathcal{L}[te^{4t}] = \int_0^{\infty} te^{4t} e^{-st} dt$$

$$= \int_0^{\infty} t e^{-t(s-4)} dt$$

$$= \left( -\frac{t}{(s-4)} e^{-(s-4)t} - \frac{1}{(s-4)^2} e^{-(s-4)t} \right) \Big|_0^{\infty}$$

$$= 0 - 0 + 0 + \frac{1}{(s-4)^2} e^0, s > 4$$

$$\boxed{\mathcal{L}[te^{4t}] = \frac{1}{(s-4)^2}}$$

Int. By Parts

$u$	$dv$
$t$	$e^{-(s-4)t}$
$1$	$-\frac{1}{(s-4)} e^{-(s-4)t}$
$0$	$\frac{1}{(s-4)^2} e^{-(s-4)t}$

2. Find the Laplace transform of the following functions:

a.  $f(t) = \sin^2(t) + \cos^2(t)$

$$f(t) = 1 = e^{0t}$$

$$\boxed{\mathcal{L}[f(t)] = F(s) = \frac{1}{s}}$$

b.  $g(t) = U(t-4)(t^2 - 8t + 16) = u_4(t)(t^2 - 8t + 16)$

$$= u_4(t)(t-4)^2, \quad f(t-4) = (t-4)^2, \quad f(t) = t^2$$

$$\mathcal{L}[g(t)] = G(s) = e^{-4s} \mathcal{L}[t^2]$$

$$= e^{-4s} \left( \frac{2!}{s^{2+1}} \right) = \boxed{e^{-4s} \left( \frac{2}{s^3} \right)}$$

c.  $h(t) = U(t-1)(t^2) = u_1(t)(t^2), \quad g(t) = t^2, \quad g(t+1) = (t+1)^2$

$$\mathcal{L}[h(t)] = H(s) = e^{-s} \mathcal{L}[g(t+1)]$$

$$= e^{-s} \mathcal{L}[(t+1)^2]$$

$$= e^{-s} \mathcal{L}[t^2 + 2t + 1]$$

$$= e^{-s} \mathcal{L}[t^2] + 2e^{-s} \mathcal{L}[t] + e^{-s} \mathcal{L}[1] \quad (1 = e^{0t})$$

$$= e^{-s} \left( \frac{2!}{s^3} \right) + 2e^{-s} \left( \frac{1!}{s^2} \right) + e^{-s} \left( \frac{1}{s} \right)$$

$$\boxed{e^{-s} \left( \frac{2}{s^3} \right) + 2e^{-s} \left( \frac{1}{s^2} \right) + e^{-s} \left( \frac{1}{s} \right)}$$

3. Find the inverse Laplace transform of the following:

a.  $F(s) = \frac{4s+6}{s^2+4} = \frac{4s}{s^2+2^2} + \frac{6}{s^2+2^2} = 4 \left( \frac{s}{s^2+2^2} \right) + \frac{6}{2} \left( \frac{2}{s^2+2^2} \right)$

$$\mathcal{L}^{-1}[F(s)] = f(t) = 4 \cos(2t) + 3 \sin(2t)$$

b.  $G(s) = \frac{s+2}{s^2+2s+5} = \frac{s+2}{(s+1)^2+4} = \frac{s+1}{(s+1)^2+4} + \frac{1}{(s+1)^2+4} = \frac{s+1}{(s+1)^2+2^2} + \frac{1}{2} \left( \frac{2}{(s+1)^2+2^2} \right)$

$b^2 - 4ac = 4 - 20 < 0$

Complete the square:

$$s^2+2s+5 = s^2+2s + \left(\frac{2}{2}\right)^2 + 5 - \left(\frac{2}{2}\right)^2 = (s+1)^2 + 4$$

c.  $H(s) = \frac{4s^2+s}{s^3(s+1)^2} = \frac{s(4s+1)}{s^3(s+1)^2} = \frac{4s+1}{s^2(s+1)^2}$

Partial Fractions

$$\frac{4s+1}{s^2(s+1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2}$$

$$As(s+1)^2 + B(s+1)^2 + Cs^2(s+1) + Ds^2 = 4s+1$$

$$As(s^2+2s+1) + B(s^2+2s+1) + Cs^3+s^2 + Ds^2 = 4s+1$$

$$\begin{aligned} A+C &= 0 & 2A+B+C+D &= 0 & A+2B &= 4 & B &= 1 \\ A &= -C & 4+1-2+D &= 0 & A &= 4-2B & & \\ C &= -2 & D &= -3 & A &= 2 & & \end{aligned}$$

d.  $J(s) = e^{-2s} \frac{16}{s(s^2+4s+8)} = e^{-2s} \left( \frac{16}{s(s^2+4s+8)} \right)$

$b^2 - 4ac = 16 - 32 < 0$

Complete the square:

$$s^2+4s+8 = s^2+4s + \left(\frac{4}{2}\right)^2 + 8 - \left(\frac{4}{2}\right)^2 = (s+2)^2 + 4$$

Partial Fractions:

$$\frac{16}{s(s^2+4s+8)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+8}$$

$$A(s^2+4s+8) + s(Bs+C) = 16$$

$$A+B=0 \quad 4A+C=0 \quad 8A=16$$

$$B=-A \quad C=-4A \quad A=2$$

$$B=-2 \quad C=-8$$

$$\begin{aligned} \mathcal{L}^{-1}[G(s)] &= g(t) \\ &= e^{-t} \cos(2t) \\ &\quad + \frac{1}{2} e^{-t} \sin(2t) \end{aligned}$$

$$\begin{aligned} H(s) &= \frac{2}{s} + \frac{1}{s^2} - \frac{2}{s+1} - \frac{3}{(s+1)^2} \\ &= 2\left(\frac{1}{s}\right) + \frac{1}{s^2} - 2\left(\frac{1}{s+1}\right) - 3\left(\frac{1}{(s+1)^2}\right) \end{aligned}$$

$$\begin{aligned} \mathcal{L}^{-1}[H(s)] &= h(t) \\ &= 2 + t - 2e^{-t} - 3e^{-t}t \end{aligned}$$

$$J(s) = e^{-2s} \left( \frac{2}{s} - \frac{2s+8}{(s+2)^2+4} \right)$$

$$= 2e^{-2s} \left( \frac{1}{s} \right) - 2e^{-2s} \left( \frac{s+4}{(s+2)^2+2^2} \right)$$

$$= 2e^{-2s} \left( \frac{1}{s} \right) - 2e^{-2s} \left( \frac{s+2}{(s+2)^2+2^2} \right)$$

$$- 2e^{-2s} \left( \frac{2}{(s+2)^2+2^2} \right)$$

$$\mathcal{L}^{-1}[J(s)] = j(t)$$

$$= 2u_2(t) - 2u_2(t)e^{-2(t-2)} \cos(2(t-2))$$

$$- 2u_2(t)e^{-2(t-2)} \sin(2(t-2))$$

4. Solve the initial value problem:  $y'' + y = 2e^t, y(0) = 1, y'(0) = -2$

a. Using Laplace transforms

$$\begin{aligned} \mathcal{L}[y''] + \mathcal{L}[y] &= 2\mathcal{L}[e^t] \\ s^2 \mathcal{L}[y] - s(1) - (-2) + \mathcal{L}[y] &= 2\left(\frac{1}{s-1}\right) \\ (s^2+1)\mathcal{L}[y] &= 2\left(\frac{1}{s-1}\right) + s-2 \\ \mathcal{L}[y] &= \frac{2}{(s-1)(s^2+1)} + \frac{s-2}{s^2+1} = \frac{1}{s-1} - \frac{s+1}{s^2+1} + \frac{s-2}{s^2+1} \\ &= \frac{1}{s-1} - \frac{3}{s^2+1} \\ &= \frac{1}{s-1} - 3\left(\frac{1}{s^2+1^2}\right) \end{aligned}$$

Partial Fractions:

$$\begin{aligned} \frac{2}{(s-1)(s^2+1)} &= \frac{A}{s-1} + \frac{Bs+C}{s^2+1} \\ A(s^2+1) + (Bs+C)(s-1) &= 2 \\ A(s^2+1) + Bs^2 + Cs - Bs - C &= 2 \\ A+B &= 0 & C-B &= 0 & A-C &= 2 \\ B &= -A & C &= B = -A & A+A &= 2 \\ B &= -1 & C &= -1 & A &= 1 \end{aligned}$$

$$y = \mathcal{L}^{-1}[\mathcal{L}[y]] = e^t - 3\sin(t)$$

b. Using the Method of Undetermined Coefficients

$$\begin{aligned} y_h: y_h'' + y_h &= 0, y_h = e^{rt} \\ r^2 + 1 &= 0 \\ r^2 &= -1, r = \pm i \\ e^{rt} &= e^{\pm it} = \cos t + i\sin t \\ y_h &= k_1 \cos t + k_2 \sin t \end{aligned}$$

$$\begin{aligned} y_p: y_p'' + y_p &= 2e^t \\ y_p &= \alpha e^t, y_p' = \alpha e^t \\ y_p'' &= \alpha e^t \\ \alpha e^t + \alpha e^t &= 2e^t \\ 2\alpha &= 2, \alpha = 1 \\ y_p &= e^t \end{aligned}$$

$$y(t) = k_1 \cos t + k_2 \sin t + e^t, y(0) = k_1 + 1 = 1, k_1 = 0, y'(0) = k_2 + 1 = -2, k_2 = -3$$

5. Solve the initial value problem  $2y' + 6y = 36U(t-1), y(0) = 1$

$$2\mathcal{L}[y'] + 6\mathcal{L}[y] = 36\mathcal{L}[u_1(t)]$$

$$2(s\mathcal{L}[y] - 1) + 6\mathcal{L}[y] = 36\left(\frac{e^{-s}}{s}\right)$$

$$(2s+6)\mathcal{L}[y] = e^{-s}\left(\frac{36}{s}\right) + 2$$

$$\begin{aligned} \mathcal{L}[y] &= e^{-s}\left(\frac{18}{s(s+3)}\right) + \frac{1}{s+3} = e^{-s}\left(\frac{6}{s} - \frac{6}{s+3}\right) + \frac{1}{s+3} \\ &= 6e^{-s}\left(\frac{1}{s}\right) - 6e^{-s}\left(\frac{1}{s+3}\right) + \frac{1}{s+3} \end{aligned}$$

Partial Fractions:

$$\frac{18}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$$

$$A(s+3) + Bs = 18$$

$$s = -3: -3B = 18, B = -6, \quad s = 0: 3A = 18, A = 6$$

$$y(t) = 6u_1(t) - 6u_1(t)e^{-3(t-1)} + e^{-3t}$$

6. Solve the initial value problem  $y'' + 2y' + 9y = 1 - u_2(t) + \delta_3(t)$ ,  $y(0) = 0, y'(0) = 1$

$$\mathcal{L}[y''] + 2\mathcal{L}[y'] + 9\mathcal{L}[y] = \mathcal{L}[1] - \mathcal{L}[u_2(t)] + \mathcal{L}[\delta_3(t)]$$

$$s^2 \mathcal{L}[y] - s(0) - 1 + 2(s \mathcal{L}[y] - 0) + 9\mathcal{L}[y] = \frac{1}{s} - \frac{e^{-2s}}{s} + e^{-3s}$$

$$(s^2 + 2s + 9)\mathcal{L}[y] = \frac{1}{s} - \frac{e^{-2s}}{s} + e^{-3s} + 1$$

$$\mathcal{L}[y] = \frac{1}{s(s^2 + 2s + 9)} - \frac{e^{-2s}}{s(s^2 + 2s + 9)} + \frac{e^{-3s}}{s^2 + 2s + 9} + \frac{1}{s^2 + 2s + 9}$$

Partial Fractions:

$$\frac{1}{s(s^2 + 2s + 9)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 9}$$

$$A(s^2 + 2s + 9) + s(Bs + C) = 1$$

$$A + B = 0 \quad 2A + C = 0 \quad 9A = 1$$

$$B = -A \quad C = -2A \quad A = \frac{1}{9}$$

$$B = -\frac{1}{9} \quad C = -\frac{2}{9}$$

Complete The Square

$$s^2 + 2s + 9 = s^2 + 2s + \left(\frac{2}{2}\right)^2 + 9 - \left(\frac{2}{2}\right)^2 \\ = (s+1)^2 + 8$$

$$\mathcal{L}[y] = \frac{\frac{1}{9}}{s} - \frac{\frac{1}{9}s + \frac{2}{9}}{(s+1)^2 + 8}$$

$$-e^{-2s} \left(\frac{\frac{1}{9}}{s}\right) + e^{-2s} \left(\frac{\frac{1}{9}s + \frac{2}{9}}{(s+1)^2 + 8}\right)$$

$$+ e^{-3s} \left(\frac{1}{(s+1)^2 + 8}\right) + \frac{1}{(s+1)^2 + 8}$$

$$\mathcal{L}[y] = \frac{1}{9} \left(\frac{1}{s}\right)$$

$$- \frac{1}{9} \left(\frac{s+2}{(s+1)^2 + 8}\right) - \frac{1}{9} e^{-2s} \left(\frac{1}{s}\right)$$

$$+ \frac{1}{9} e^{-2s} \left(\frac{s+2}{(s+1)^2 + 8}\right) + e^{-3s} \left(\frac{1}{(s+1)^2 + 8}\right)$$

$$+ \frac{1}{(s+1)^2 + 8}$$

$$\mathcal{L}[y] = \frac{1}{9} \left(\frac{1}{s}\right) - \frac{1}{9} \left(\frac{s+1}{(s+1)^2 + 8}\right) - \frac{1}{9} \left(\frac{1}{\sqrt{8}}\right) \left(\frac{\sqrt{8}}{(s+1)^2 + 8}\right)$$

$$- \frac{1}{9} e^{-2s} \left(\frac{1}{s}\right) + \frac{1}{9} e^{-2s} \left(\frac{s+1}{(s+1)^2 + 8}\right) + \frac{1}{9} e^{-2s} \left(\frac{1}{\sqrt{8}}\right) \left(\frac{\sqrt{8}}{(s+1)^2 + 8}\right)$$

$$+ e^{-3s} \left(\frac{1}{\sqrt{8}}\right) \left(\frac{\sqrt{8}}{(s+1)^2 + 8}\right) + \frac{1}{\sqrt{8}} \left(\frac{\sqrt{8}}{(s+1)^2 + 8}\right)$$

$$y(t) = \frac{1}{9} - \frac{1}{9} e^{-t} \cos(\sqrt{8}t) + \frac{8}{9\sqrt{8}} e^{-t} \sin(\sqrt{8}t)$$

$$- \frac{1}{9} u_2(t) + \frac{1}{9} u_2(t) e^{-(t-2)} \cos(\sqrt{8}(t-2)) + \frac{1}{9\sqrt{8}} u_2(t) e^{-(t-2)} \sin(\sqrt{8}(t-2))$$

$$+ \frac{1}{\sqrt{8}} u_3(t) e^{-(t-3)} \sin(\sqrt{8}(t-3)) + \frac{1}{\sqrt{8}} e^{-t} \sin(\sqrt{8}t)$$