Introduction

Advanced Engineering Mathematics

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Overview/Keywords/References

Advanced Engineering Mathematics | **Slide Set Zero**

<u>Assumptions and Objectives</u>

Reference Text: EK 7-8, 11-12 | **Example: N/A**

- See Also:
  - Lecture Notes : 00.OverviewAndOutline.pdf
- Start:
  - Lecture Notes : 01.LinearDefinitions.pdf
  - EK.7.1-7.2
  - EK.7.3, EK.7.5
- Finish:
  - *Break*

Before We Begin

Quote of Slide Set Zero
Hank: They can either paint it, or draw it, or write it down and then pass it on to somebody. They read what you're saying and then they reexperience. That's the only connection you have with them. You can't rewrite. To rewrite is to deceive and lie and you betray your own thoughts. **Martin**: I can't accept your interpretation of my, necessity to rewrite every single word. Guilt is the key, not sin. Guilt me not writing the best that I can, considering everything from every possible angle. Balancing everything.
Naked Lunch (film) : David Cronenberg (1991)

Assumptions : MATH225 - Part I

- First-order ODE : $y' = f(y,t)$
  - Integrating Factors or Undetermined Coefficients (UC) : $f(y,t) = a(t)y + b(t)$
  - Separation of Variables : $f(y,t) = h(t)g(y)$
  - Phase-Space Analysis : $f(y,t) = g(y)$
- Second-order Linear ODE: $a(t)y'' + b(t)y' + c(t)y = f(t)$
  - General Solution : $y(t) = y_h(t) + y_p(t)$
  - If $y_1 y_2' - y_1' y_2 \neq 0$ for some initial condition then $y_h(t) = c_1 y_1(t) + c_2 y_2(t)$.
  - UC when $a, b, c \in \mathbb{R}$ and $f(t)$ is expressible as a product and/or sum of polynomial and exponential functions.
  - Power-series : $a(t), b(t), c(t) \in C^2$

Assumptions : MATH225 - Part II

- Beats : $a = 1, b = 0, c = \omega^2$ and $f(t + \delta w) = f(t)$ where δ is small on some timescale.
  - $\lim_{t \to \infty} y(t) < \infty$
- Resonance : $a = 1, b = 0, c = \omega^2$ and $f(t + w) = f(t)$.
  - $\lim_{t \to \infty} y(t) = \infty$
- Eigenfunction Decomposition : $\mathbf{Y}' = \mathbf{AY}, \ \mathbf{A} \in \mathbb{R}^{n \times n}, \ n \in \mathbb{N}$
  - Assuming there is a diagonal $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$ leads to the problem $\tilde{\mathbf{Y}}' = \mathbf{D}\tilde{\mathbf{Y}}$ whose solution is given by, $\mathbf{Y}(t) = \sum_{i=1}^{n} c_i \mathbf{Y}_i e^{\lambda_i t}$.

Assumptions : Calculus - Part I

- Single Variable Calculus
  - Limits and Continuity : $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = f(x_0)$ for all $x \in \mathbb{R} \Rightarrow f \in C(\mathbb{R})$
  - Fundamental Theorem of Calculus : $\dfrac{d}{dx} \displaystyle\int_{x_0}^{x} f(t)dt = f(x),\ x_0 < x$
- Taylor Series:

$$
\begin{aligned}
e^{(a+bi)x} = e^{ax} e^{ibx} &= \sum_{n=0}^{\infty} \frac{a^n x^n}{n!} \sum_{m=0}^{\infty} \frac{i^m b^m x^m}{m!} \\
&= \sum_{n=0}^{\infty} \frac{a^n x^n}{n!} \left(\sum_{m=0}^{\infty} \frac{(-1)^m (bx)^{2m}}{2m!} + i \sum_{m=0}^{\infty} \frac{(-1)^m (bx)^{2m+1}}{(2m+1)!} \right) \\
&= e^{ax} \left(\cos(bx) + i \sin(bx) \right)
\end{aligned}
$$

Calculus - Part II

- Multivariate Chain Rule : $y_i = y_i(x_1, x_2)$
  - $\partial_{x_i} u(y_1, y_2) = \partial_{y_1} u \, \partial_{x_i} y_1 + \partial_{y_2} u \, \partial_{x_i} y_2$ for $i = 1, 2$
- Vector Calculus : $\nabla = \langle \partial_x, \partial_y, \partial_z \rangle = [\partial_x, \partial_y, \partial_z]^{\mathsf{T}}$
  - $\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$
  - $\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$
- Stokes Theorem : $\displaystyle\int_M \mathrm{d}\boldsymbol{\omega} = \oint_{\partial M} \boldsymbol{\omega}$
  - A flux integral of a vector field over its boundary is equivalent to a volume integral of a differential form of the vector field.
  - $\displaystyle\iiint_V \text{div}(\mathbf{F}) dV = \iint_{\partial V} \mathbf{F} \cdot \hat{\mathbf{n}} dA$

Assumptions : ?

- You have taken CSM Core
- You are majoring in any one of the following:
  - EG
  - GPGH
  - MATH
  - Econ?
- You may have taken or are now in:
  - Info Systems
  - Linear Algebra
  - Static Fields
  - Physics III
  - PDE

Course Description : Bulletin

MATH348: Introduction to partial differential equations, with applications to physical phenomena. Fourier series. Linear algebra, with emphasis on sets of simultaneous equations.

- Similar Courses:
  - Boston University
  - The Chinese University of Hong Kong : ERG2011A Advanced Engineering Maths
  - The University of Texas at Dallas : EE 3300-001 Advanced Engineering Mathematics
  - *MIT*

Goals and Objectives

- Goals:
  - Study the concepts of mathematical modeling through the techniques provided by the bulletin.
  - Understand linear mathematics as it pertains to Algebra and Differential Equations.
- Objectives:
  - Linear Systems of Equations : $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$
  - Linear Scalings : $\mathbf{Ax} = \lambda \mathbf{x}$ where $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\lambda \in \mathbb{C}$.
  - Plancherel theorem : $\mathfrak{F}_p : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the unique isometry onto L^2.
  - Linear PDE : Elliptic, $\triangle u = f$, Parabolic, $u_t = c^2 \triangle u + f$, and Hyperbolic, $u_{tt} = c^2 \triangle u + f$ PDE.

Linear Systems of Algebraic Equations

- Key Point:
  - Problems involving systems of linear equations can be algebraically manipulated as matrices and vectors.
- Key Question:
  - Given $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^{m}$. When does the solution to $\mathbf{Ax} = \mathbf{b}$ exist? Also, if a solution exists then when is this solution unique?
- Key Answer:
  - If $\mathbf{b}$ is in the *column space* of $\mathbf{A}$ then a solution exists. If the dimension of the *null space* of $\mathbf{A}$ is zero then this solution is unique.

Spectrum and Basis

- Linear Scalings are Common:
  - Exponential Growth/Decay : $y' = \alpha y$, $\alpha \in \mathbb{R}$
  - Constant Linear ODE : $\mathbf{Y}' = \mathbf{A}\mathbf{Y}$, $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{Y}(0) \in \mathbb{R}^n$
  - Linear PDE : $\triangle u = \lambda u$, $\lambda \in \mathbb{R}$
- For martix equations, linear algebra asks:
  - Find all $\lambda \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^n$ such that $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$.
- The key tasks are to solve the following:
  - Roots of Characteristic Polynomial : $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$
  - Basis for each Null Space : $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$
- Almost all matrices admit an eigenbasis for the domain of transformation by **A**:
  - Diagonalization : $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, $[\mathbf{D}]_{ij} = d_i \delta_{ij}$

Fourier Transform (FT)

- FT : Let $f, \hat{f} \in L^1(\mathbb{R})$ then
$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \omega x} dx \iff f(x) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{2\pi i \omega x} d\omega$$
  - Parseval's Theorem : $||f|| = ||\hat{f}||$
  - Uncertainty Relation : $16\pi^2 ||x^2 f^2|| \; ||\omega^2 \hat{f}^2|| \geq 1$
- Plancherel Theorem : Using the Hermite polynomials, defined by, $H'' - xH' = \lambda H$, it is possible to extend these results to the Hilbert space $L^2(\mathbb{R})$.
- Fourier Series :
$$\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} c_n \delta\left(\omega - \frac{n}{2\pi}\right) \iff f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Partial Differential Equations

- Given $\alpha u_{tt} + \beta u_t = c^2 \triangle u + f$, where $u = u(\mathbf{x}, t), \;\; \mathbf{x} \in D \subset \mathbb{R}^3, \; t \in (0, \infty)$.
  - Laplacian : $\triangle u = u_{xx} + u_{yy} + u_{zz}$
  - Heat/Diffusion : $\alpha = 0$ and $\beta \neq 0$
  - Wave : $\alpha \neq 0$ and $\beta = 0$
  - Homogeneous : $f = f(\mathbf{x}, t) = 0$
  - Dirichlet : $u(\mathbf{x}, t) = 0$ for $x \in \partial D$
  - Neumann : $u_{x_i}(\mathbf{x}, t) = 0$ for $x \in \partial D$
- Suppose we have a homogeneous heat problem in $\mathbb{R}^{1+1}$ with Dirchlet boundary conditions. If the object has finite length π then,

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 c^2 t}$$

Conclusions

- Linear mathematics obeys an algebraic structure that simplifies analysis in any context. In many ways it is a 'complete' theory. That is, if a problem is linear then its general solution can always be expressed as a linear combination of elementary solutions.
- The concept of a linear vector space allows one to define the mathematical construct where solutions to linear problems 'live.' If a basis for this space is defined then ANY element of that space can be defined by linear combination.
- Particular types of functions also satisfy the vector space axioms and define a linear function space, which abstracts the notion of $\mathbb{R}^n$ and has been used to connect linear problems from various areas of mathematics.