Advanced Engineering Mathematics		**Homework Eight Solutions**
Partial Differential Equations : Heat Equation, Wave Equation, Properties, External Forcing		
Text: 12.3-12.5	Lecture Notes : 14 and 15	Lecture Slides: 6

Quote of Homework Eight Solutions
Go down in your own way and everyday is the right day and as you rise above the fearlines in his frown you look down and hear the sound of the faces in the crowd.
Pink Floyd : Fearless - Meddle (1971)

1. Heat Equation on a closed and bounded spatial domain of $\mathbb{R}^{1+1}$

Consider the one-dimensional heat equation,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad , \tag{1}$$

$$x \in (0, L)\,, \qquad t \in (0, \infty)\,, \qquad c^2 = \frac{K}{\sigma\rho}. \tag{2}$$

Equations (1)-(2) model the time-evolution of the temperature, $u = u(x,t)$, of a heat conducting medium in one-dimension. The object, of length L, is assumed to have a homogenous thermal conductivity K, specific heat σ, and linear density ρ. That is, $K, \sigma, \rho \in \mathbb{R}^+$. If we consider an object of finite-length, positioned on say $(0, L)$, then we must also specify the boundary conditions[1],

$$u_x(0,t) = 0, u_x(L,t) = 0, . \tag{3}$$

Lastly, for the problem to admit a unique solution we must know the initial configuration of the temperature,

$$u(x,0) = f(x). \tag{4}$$

1.1. **Separation of Variables : General Solution.** Assume that the solution to (1)-(2) is such that $u(x,t) = F(x)G(t)$ and use separation of variables to find the general solution to (1)-(2), which satisfies (3)-(4). [2]

(1) Assume that, $u(x,t) = F(x)G(t)$ then $u_{xx} = F''(x)G(t)$ and $u_t = F(x)G'(t)$ and the 1-D heat equation becomes,

$$\frac{G'(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = -\lambda, \tag{5}$$

where we have introduced the separation constant λ.[3] From this equation we have the two ODE's,

$$G'(t) + \lambda c^2 G(t) \;=\; 0, \tag{6}$$

$$F''(x) + \lambda F(x) \;=\; 0. \tag{7}$$

Each of these ODE's can be solved through 'elementary methods' to get,[4]

$$\lambda \in \mathbb{R} \;:\; G(t) = Ae^{-\lambda c^2 t}, \quad A \in \mathbb{R}, \tag{8}$$

$$\lambda \in \mathbb{R}^+ \;:\; F(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x), \tag{9}$$

$$\lambda \in \mathbb{R}^- \;:\; F(x) = c_3 \cosh(\sqrt{|\lambda|}x) + c_4 \sinh(\sqrt{|\lambda|}x), \tag{10}$$

$$\lambda = 0 \;:\; F(x) = c_5 + c_6 x. \tag{11}$$

Each of the functions $F(x)$ must also satisfy the boundary conditions, $u_x(0,t) = 0$ and $u_x(L,t)$ and so we won't need all of them. Notice that the boundary conditions imply that,

$$u_x(0,t) \;=\; F'(0)G(t) = 0, \tag{12}$$

$$u_x(L,t) \;=\; F'(L)G(t) = 0, \tag{13}$$

[1] Here the boundary conditions correspond to perfect insulation of both endpoints

[2] An insulated bar is discussed in examples 4 and 5 on page 557.

[3] This occurs in conjunction with the following argument. Since (5) must be true for all (x,t) then both sides must be equal to a function that has neither t's nor x's. Hence they must be equal to a constant function. To see that this is true put an x or t on the side that has λ and test points.

[4] These elementary methods are those you learned in ODE's and can be found in the ODE review of the lecture slides on separation of variables.

which gives $F'(0) = 0$ and $F'(L) = 0$.[5] So, we now have to determine, which of the previous functions, $F(x)$, satisfy these boundary conditions. To this end we have the following arguments,

$$\begin{aligned}
\lambda \in \mathbb{R}^+ \quad &: \quad F'(0) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}0) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}0) = c_1\sqrt{\lambda}\cdot 0 + c_2\sqrt{\lambda}\cdot 1 \implies c_2 = 0,\\
\lambda \in \mathbb{R}^+ \quad &: \quad F'(L) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}L) = c_1\sqrt{\lambda}\cdot\sin(\sqrt{\lambda}L) + 0\cdot\sqrt{\lambda}\cos(\sqrt{\lambda}L) \implies\\
&\implies c_1\sqrt{\lambda}\cdot\sin(\sqrt{\lambda}L) = 0 \iff c_1 = 0 \ \underline{\text{or}}\ \sin(\sqrt{\lambda}L) = 0.
\end{aligned}$$

If we consider the case that $c_1 = 0$ then we have $F(x) = 0$ for $\lambda \in \mathbb{R}^+$ but we should try to keep as many solutions as possible and we ignore this case. Thus assume that $c_1 \neq 0$ we have that $\sin(\sqrt{\lambda}L) = 0$, which is true for $\sqrt{\lambda} = n\pi/L$ and we have the following eigenvalue/eigenfunction pairs indexed by n,

$$F_n(x) = c_n\cos(\sqrt{\lambda}x), \quad \lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, 3, \ldots. \tag{14}$$

We now consider the $\lambda \in \mathbb{R}^-$ case to find that,

$$\begin{aligned}
\lambda \in \mathbb{R}^- \quad &: \quad F'(x) = c_3\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}0) + c_4\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}0) = c_3\cdot 0 + c_4\cdot 1 = 0 \implies c_4 = 0,\\
\lambda \in \mathbb{R}^- \quad &: \quad F'(x) = c_3\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}L) + c_4\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}L) = c_3\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}L) + 0\cdot\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}L) =\\
&= c_3\sqrt{|\lambda|}\frac{e^{\sqrt{|\lambda|}L} - e^{-\sqrt{|\lambda|}L}}{2} = 0 \implies c_3 = 0,
\end{aligned}$$

which means that for $\lambda \in \mathbb{R}^-$ we only have the trivial solution $F(x) = 0$. Lastly, we consider the case $\lambda = 0$ to get,

$$\lambda = 0 \quad : \quad F'(0) = F'(L) = c_6 = 0 \implies c_5 \in \mathbb{R}, \tag{15}$$

which gives the last eigenpair,[6]

$$F_0 = c_0 \in \mathbb{R} \quad \lambda_0 = 0. \tag{16}$$

Noting that there are infinitely many λ's implies now that there are infinitely many temporal solutions (8) and we have,

$$G_n(t) = A_n e^{\lambda_n c^2 t}, \quad \lambda = \frac{n^2\pi^2}{L^2},\ n = 1, 2, 3, \ldots. \tag{17}$$

For the case where $\lambda = 0$ we have the ODE $G'(t) = 0$, whose solution is$G_0(t) = A_0 \in \mathbb{R}$. Thus we have infinitely many functions, that solve the PDE, of the form:

$$u_n(x,t) = F_n(x)G_n(t),\ n = 0, 1, 2, 3, \ldots \tag{18}$$

Hence since the PDE is linear superposition implies that we have the general solution,

$$\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) &(19)\\
&= F_0(x)G_0(x) + \sum_{n=1}^{\infty} F_n(x)G_n(t) &(20)\\
&= c_0\cdot A_0 + \sum_{n=1}^{\infty} A_n c_n\cos(\sqrt{\lambda_n}x)e^{\lambda_n c^2 t} &(21)\\
&= a_0 + \sum_{n=1}^{\infty} a_n\cos(\sqrt{\lambda_n}x)e^{\lambda_n c^2 t}, &(22)
\end{aligned}$$

which is the general solution of the heat equation with the given boundary conditions.

1.2. **Qualitative Dynamics.** Describe how the long term behavior of the general solution to (1)-(4) changes as the thermal conductivity, K, is increased while all other parameters are held constant. Also, describe how the solution changes when the linear density, ρ, is increased while all other parameters are held constant.

- If κ(thermal conductivity) is increased, the temporal solution decays faster and the system reaches equilibrium sooner.
- If ρ(density) is increased, the temporal solution decays slower and the system takes longer to reach equilibrium.

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[5] We assume that $G(t) = 0$ because if it did then we would have $u(x,t) = F(x)G(t) = F(x)\cdot 0 = 0$, which is called the trivial solution and is ignored since it is already in thermal equilibrium. We care about dynamics!

[6] Here we have used the subscripts to denote that these are all associated with the $\lambda = 0$ case. We have also trivially changed c_5 to c_0.

1.3. **Fourier Series : Solution to the IVP.** Define,

$$f(x) = \begin{cases} \frac{2k}{L}x, & 0 < x \le \frac{L}{2}, \\ \frac{2k}{L}(L - x), & \frac{L}{2} < x < L \end{cases} \tag{23}$$

and for the following questions we consider the solution, u, to the heat equation given by, (1)-(2), which satisfies the initial condition given by (53). [7] For $L = 1$ and $k = 1$, find the particular solution to (1)-(2) with boundary conditions (3)-(4) for when the initial temperature profile of the medium is given by (53). Show that $\lim_{t\to\infty} u(x,t) = f_{avg} = 0.5$.[8]

To find the unknown constants present in the general solution we must apply an initial condition, $u(x, 0) = f(x)$. Doing so gives,

$$u(x,0) = f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\sqrt{\lambda_n}x)e^{\lambda_n c^2 \cdot 0} \tag{24}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos(\sqrt{\lambda_n}x), \tag{25}$$

which is a Fourier cosine half-range expansion of the initial condition. Thus the unknown constants are Fourier coefficients and,

$$a_0 = \frac{1}{L}\int_0^L f(x)dx, \tag{26}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(\sqrt{\lambda_n}x)dx. \tag{27}$$

If we note that these integrals have been done in a previous homework on Fourier series, then we known these Fourier coefficients as,

$$a_0 = \frac{k}{2}, \tag{28}$$

$$a_n = \frac{4k}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}x\right) - (-1)^n - 1\right]. \tag{29}$$

Moreover, if we take $L = k = 1$ we see that $\lim_{t\to\infty} u(x,t) = a_0 = .5$, which implies that under these insulating boundary conditions the equilibrium state for the medium is a constant function $u = .5$ and that this is nothing more than the average of the initial configuration.

2. Wave Equation on a closed and bounded spatial domain of $\mathbb{R}^{1+1}$

Consider the one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \tag{30}$$

$$x \in (0, L), \quad t \in (0, \infty), \quad c^2 = \frac{T}{\rho}. \tag{31}$$

Equations (30)-(31) model the time-evolution of the displacement,$u = u(x,t)$, from rest, of an elastic medium in one-dimension. The object, of length L, is assumed to have a homogeneous lateral tension T, and linear density ρ. That is, $T, \rho \in \mathbb{R}^+$. Assume, as well, the boundary conditions[9],

$$u_x(0,t) = 0, u_x(L,t) = 0, \tag{32}$$

and initial conditions,

$$u(x,0) = f(x), \tag{33}$$

$$u_t(x,0) = g(x). \tag{34}$$

[7] When solving the following problems it would be a good idea to go back through your notes and the homework looking for similar calculations.

[8] It is interesting here to note that though the initial condition f doesn't appear to satisfy the boundary conditions its periodic Fourier extension does. That is, if you draw the even periodic extension of the initial condition then you would see that the slope is not well defined at the end points. Remembering that the Fourier series averages the right and left hand limits of the periodic extension of the function f at the endpoints shows that the boundary conditions are, in fact, satisfied, since the derivative of an average is the average of derivatives.

[9] These boundary conditions imply that the object must have zero slope at each endpoint.

2.1. **Separation of Variables : General Solution.** Assume that the solution to (30)-(31) is such that $u(x,t) = F(x)G(t)$ and use separation of variables to find the general solution to (30)-(31), which satisfies (32)-(34). [10] [11]

The only difference between this problem and the heat problem above are the time-dynamics specified by the PDE. This gives a second-order constant linear ODE in time and from this ODE we have oscillations of Fourier modes instead of exponential decay.

Assume that, $u(x,t) = F(x)G(t)$ then $u_{xx} = F''(x)G(t)$ and $u_{tt} = F(x)G''(t)$ and the 1-D wave equation becomes,

$$\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = -\lambda, \tag{35}$$

where we have introduced the separation constant λ.[12] From this equation we have the two ODE's,

$$G''(t) + \lambda c^2 G(t) = 0, \tag{36}$$

$$F''(x) + \lambda F(x) = 0. \tag{37}$$

Each of these ODE's can be solved through 'elementary methods' to get,[13]

$$\lambda \in \mathbb{R} \quad : \quad G(t) = A_1 \cos\left(c\sqrt{\lambda}t\right) + A_1^* \sin\left(c\sqrt{\lambda}t\right), \quad A_1, A_1^* \in \mathbb{R}, \tag{38}$$

$$\lambda \in \mathbb{R}^+ \quad : \quad F(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x), \tag{39}$$

$$\lambda \in \mathbb{R}^- \quad : \quad F(x) = c_3 \cosh(\sqrt{|\lambda|}x) + c_4 \sinh(\sqrt{|\lambda|}x), \tag{40}$$

$$\lambda = 0 \quad : \quad F(x) = c_5 + c_6 x. \tag{41}$$

Each of the functions $F(x)$ must also satisfy the boundary conditions, $u_x(0,t) = 0$ and $u_x(L,t)$ and so we won't need all of them. Notice that the boundary conditions imply that,

$$u_x(0,t) = F'(0)G(t) = 0, \tag{42}$$

$$u_x(L,t) = F'(L)G(t) = 0, \tag{43}$$

which gives $F'(0) = 0$ and $F'(L) = 0$.[14] So, we now have to determine, which of the previous functions, $F(x)$, satisfy these boundary conditions. To this end we have the following arguments,

$$\begin{aligned}
\lambda \in \mathbb{R}^+ \quad &: \quad F'(0) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}0) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}0) = c_1\sqrt{\lambda}\cdot 0 + c_2\sqrt{\lambda}\cdot 1 \implies c_2 = 0,\\
\lambda \in \mathbb{R}^+ \quad &: \quad F'(L) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}L) = c_1\sqrt{\lambda}\cdot\sin(\sqrt{\lambda}L) + 0\cdot\sqrt{\lambda}\cos(\sqrt{\lambda}L) \implies\\
&\implies c_1\sqrt{\lambda}\cdot\sin(\sqrt{\lambda}L) = 0 \iff c_1 = 0 \ \underline{\text{or}}\ \sin(\sqrt{\lambda}L) = 0.
\end{aligned}$$

If we consider the case that $c_1 = 0$ then we have $F(x) = 0$ for $\lambda \in \mathbb{R}^+$ but we should try to keep as many solutions as possible and we ignore this case. Thus assume that $c_1 \neq 0$ we have that $\sin(\sqrt{\lambda}L) = 0$, which is true for $\sqrt{\lambda} = n\pi/L$ and we have the following eigenvalue/eigenfunction pairs indexed by n,

$$F_n(x) = c_n \cos(\sqrt{\lambda}x), \quad \lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, 3, \dots. \tag{44}$$

We now consider the $\lambda \in \mathbb{R}^-$ case to find that,

$$\begin{aligned}
\lambda \in \mathbb{R}^- \quad &: \quad F'(x) = c_3\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}0) + c_4\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}0) = c_3\cdot 0 + c_4 \cdot 1 = 0 \implies c_4 = 0,\\
\lambda \in \mathbb{R}^- \quad &: \quad F'(x) = c_3\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}L) + c_4\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}L) = c_3\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}L) + 0\cdot\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}L) =\\
&= c_3\sqrt{|\lambda|}\frac{e^{\sqrt{|\lambda|}L} - e^{-\sqrt{|\lambda|}L}}{2} = 0 \implies c_3 = 0,
\end{aligned}$$

which means that for $\lambda \in \mathbb{R}^-$ we only have the trivial solution $F(x) = 0$. Lastly, we consider the case $\lambda = 0$ to get,

$$\lambda = 0 \quad : \quad F'(0) = F'(L) = c_6 = 0 \implies c_5 \in \mathbb{R}, \tag{45}$$

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[10] It is important to notice that the solution to the spatial portion of the problem is the same as the heat problem above.

[11] Remember that in this case we have a nontrivial spatial solution for zero eigenvalue. From this you should find the associated temporal function should find that $G_0(t) = C_1 + C_2 t$.

[12] This occurs in conjunction with the following argument. Since (35) must be true for all (x,t) then both sides must be equal to a function that has neither t's nor x's. Hence they must be equal to a constant function. To see that this is true put an x or t on the side that has λ and test points.

[13] These elementary methods are those you learned in ODE's and can be found in the ODE review of the lecture slides on separation of variables.

[14] We assume that $G(t) = 0$ because if it did then we would have $u(x,t) = F(x)G(t) = F(x)\cdot 0 = 0$, which is called the trivial solution and is ignored since it is already in thermal equilibrium. We care about dynamics!

which gives the last eigenpair,[15]

$$F_0 = c_0 \in \mathbb{R} \quad \lambda_0 = 0. \tag{46}$$

Noting that there are infinitely many λ's implies now that there are infinitely many temporal solutions (38) and we have,

$$G_n(t) = G(t) = A_n \cos\left(c\sqrt{\lambda_n} t\right) + A_n^* \sin\left(c\sqrt{\lambda_n} t\right), \quad \lambda_n = \frac{n^2\pi^2}{L^2}, \; n = 1, 2, 3, \ldots. \tag{47}$$

For the case where $\lambda = 0$ we have the ODE $G''(t) = 0$, whose solution is$G_0(t) = A_0 + A_0^* t$. Thus we have infinitely many functions, that solve the PDE, of the form:

$$u_n(x,t) = F_n(x)G_n(t), \; n = 0, 1, 2, 3, \ldots \tag{48}$$

Hence since the PDE is linear superposition implies that we have the general solution,

$$\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) && (49)\\
&= F_0(x)G_0(x) + \sum_{n=1}^{\infty} F_n(x)G_n(t) && (50)\\
&= c_0 \cdot (A_0 + A_0^* t) + \sum_{n=1}^{\infty} c_n \cos(\sqrt{\lambda_n} x)\left[A_n \cos\left(c\sqrt{\lambda_n} t\right) + A_n^* \sin\left(c\sqrt{\lambda_n} t\right)\right] && (51)\\
&= a_0 + a_0^* t + \sum_{n=1}^{\infty} \cos(\sqrt{\lambda_n} x)\left[a_n \cos\left(c\sqrt{\lambda_n} t\right) + a_n^* \sin\left(c\sqrt{\lambda_n} t\right)\right], && (52)
\end{aligned}$$

which is the general solution of the one-dimensional wave equation with the given the boundary conditions.

2.2. **Qualitative Dynamics.** Describe how the the general solution to (30)-(31) changes as the tension, T, is increased while all other parameters are held constant. Also, describe how the solution changes when the linear density, ρ, is increased while all other parameters are held constant.

- If T(Tension) is increased, the temporal solution oscillate faster.
- If ρ(density) is increased, the temporal oscillates slower.

2.3. **Fourier Series : Solution to the IVP.** Define,

$$f(x) = \begin{cases} \dfrac{2k}{L} x, & 0 < x \le \frac{L}{2}, \\[2ex] \dfrac{2k}{L}(L - x), & \frac{L}{2} < x < L. \end{cases} \tag{53}$$

Let $L = 1$ and $k = 1$ and find the particular solution, which satisfies the initial displacement, $f(x)$, given by (53) and has zero initial velocity for all points on the object.

To find the unknown constants present in the general solution we must apply an initial condition, $u(x,0) = f(x)$. Doing so gives,

$$\begin{aligned}
u(x,0) = f(x) &= a_0 + a_0^* \cdot 0 + \sum_{n=1}^{\infty} \cos(\sqrt{\lambda_n} x)\left[a_n \cos\left(c\sqrt{\lambda_n} \cdot 0\right) + a_n^* \sin\left(c\sqrt{\lambda_n} \cdot 0\right)\right] && (54)\\
&= a_0 + \sum_{n=1}^{\infty} a_n \cos(\sqrt{\lambda_n} x), && (55)
\end{aligned}$$

which is a Fourier cosine half-range expansion of the initial condition. Thus the unknown constants are Fourier coefficients and,

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_0^L f(x)dx, && (56)\\
a_n &= \frac{2}{L}\int_0^L f(x)\cos(\sqrt{\lambda_n} x)dx. && (57)
\end{aligned}$$

If we note that these integrals have been done in a previous homework on Fourier series, then we known these Fourier coefficients as,

$$\begin{aligned}
a_0 &= \frac{k}{2}, && (58)\\
a_n &= \frac{4k}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2} x\right) - (-1)^n - 1\right]. && (59)
\end{aligned}$$

[15]Here we have used the subscripts to denote that these are all associated with the $\lambda = 0$ case. We have also trivially changed c_5 to c_0.

Now we apply the initial velocity, $u_t(x,0) = g(x) = 0$ to get that,

$$u_t(x,0) = g(x) = a_0^* + \sum_{n=1}^{\infty} \cos(\sqrt{\lambda_n}x)\left[a_n c\sqrt{\lambda_n}\sin\left(c\sqrt{\lambda_n}\cdot 0\right) + a_n^* c\sqrt{\lambda_n}\cos\left(c\sqrt{\lambda_n}\cdot 0\right)\right] \tag{60}$$

$$= a_0^* + \sum_{n=1}^{\infty} a_n^* c\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x) \tag{61}$$

which is a Fourier cosine half-range expansion of the initial condition. Thus the unknown constants are Fourier coefficients and,

$$a_0^* = \frac{1}{L}\int_0^L g(x)dx, \tag{62}$$

$$a_n^* = \frac{2}{c\sqrt{\lambda_n}L}\int_0^L g(x)\cos(\sqrt{\lambda_n}x)dx, \tag{63}$$

but since $g(x) = 0$ we have that $a_0^* = a_n^* = 0$ for all n.[16]

3. D'Alembert Solution to the Wave Equation in $\mathbb{R}^{1+1}$

Show that by direct substitution that the function $u(x,t)$ given by,

$$u(x,t) = \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(y)dy, \tag{64}$$

is a solution to the one-dimensional wave equation where u_0 and v_0 are the ideally elastic objects initial displacement and velocity, respectively. [17]

We must take derivatives of $u(x,t)$. We note the following relations, which follow from the chain-rule.

$$\frac{\partial}{\partial t}\left[u_0(x\mp ct)\right] = \frac{\partial u_0}{\partial\left[x\mp ct\right]}\frac{\partial\left[x\mp ct\right]}{\partial t} = u_0'\cdot\mp c \implies \frac{\partial^2}{\partial t^2}\left[u_0(x\mp ct)\right] = c^2 u_0''(x\mp ct) \tag{65}$$

$$\frac{\partial}{\partial x}\left[u_0(x\mp ct)\right] = \frac{\partial u_0}{\partial\left[x\mp ct\right]}\frac{\partial\left[x\mp ct\right]}{\partial x} = u_0' \implies \frac{\partial^2}{\partial x^2}\left[u_0(x\mp ct)\right] = u_0''(x\mp ct) \tag{66}$$

The derivatives on the second term involving the initial-velocity are more difficult. First, note the following simplification,

$$\int_{x-ct}^{x+ct} v_0(y)dy = \int_0^{x+ct} v_0(y)dy - \int_0^{x-ct} v_0(y)dy. \tag{67}$$

Also note the following formula that follows from the fundamental theorem of calculus.

$$\frac{d}{dx}\int_0^{f(x)} g(t)dt = \frac{df}{dx}g(f(x)) \tag{68}$$

Thus,

$$\frac{\partial}{\partial t}\left[\int_0^{x\pm ct} v_0(y)dy\right] = \frac{\partial}{\partial t}\left[x\pm ct\right]v_0(x\pm ct) \tag{69}$$

$$= \pm c v_0(x\pm ct) \implies \frac{\partial^2}{\partial t^2}\left[\int_0^{x\pm ct} v_0(y)dy\right] = c^2 v_0'(x\pm ct), \tag{70}$$

and

$$\frac{\partial}{\partial x}\left[\int_0^{x\pm ct} v_0(y)dy\right] = \frac{\partial}{\partial x}\left[x\pm ct\right]v_0(x\pm ct) \tag{71}$$

$$= v_0(x\pm ct) \implies \frac{\partial^2}{\partial x^2}\left[\int_0^{x\pm ct} v_0(y)dy\right] = v_0'(x\pm ct). \tag{72}$$

With this in hand we now have,

$$u_{tt} = \frac{c^2}{2}\left[u_0''(x-ct) + u_0''(x+ct)\right] + \frac{c}{2}\left[v_0'(x+ct) - v_0'(x-ct)\right] \tag{73}$$

$$c^2 u_{xx} = c^2\left(\frac{1}{2}\left[u_0''(x-ct) + u_0''(x+ct)\right] + \frac{1}{2c}\left[v_0'(x+ct) - v_0'(x-ct)\right]\right), \tag{74}$$

which implies that $u_{tt} - c^2 u_{xx} = 0$.

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[16] It is interesting to note that if $a_0^* \neq 0$ then the displacement grows with time. This is a consequence of the boundary conditions, which require the string to be flat at the endpoints but do not require they stay fixed.

[17] This is called the D'Alembert solution to the wave equation. To do this you may want to recall the fundamental theorem of calculus, $\frac{d}{dx}\int_0^x f(t)dt = f(x)$ and properties of integrals, $\int_{-a}^a f(x)dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx$.

4. Time Dependent Boundary Conditions

It makes sense to consider time-dependent interface conditions. That is, (1) and (4) subject to

$$u(0,t) = g(t), \;\; u(L,t) = h(t), \;\; t \in (0,\infty) \tag{75}$$

Show that this PDE transforms into:

$$\frac{\partial w}{\partial t} = c^2 \frac{\partial^2 w}{\partial x^2} - S_t(x,t) \quad , \tag{76}$$

$$x \in (0,L)\,, \qquad t \in (0,\infty)\,, \qquad c^2 = \frac{\kappa}{\rho\sigma}. \tag{77}$$

with boundary conditions and initial conditions,

$$w(0,t) = w(L,t) = 0, \tag{78}$$

$$w(x,0) = F(x), \tag{79}$$

where $F(x) = f(x) - S(x,0)$ and $S(x,t) = \dfrac{h(t)+g(t)}{L}x + g(t).$ [18]

We need to find the correct function $w(x,t)$ such that (1) and (4) subject to the given boundary conditions transforms to (76)-(79). Specifically, we need to find,

$$w(x,t) = u(x,t) - T(x), \tag{80}$$

such that $w(0,t) = w(L,t) = 0$. Hoping for the best we assume that T is a linear function of x, $T(x) = ax + b$, where a and b are unknown. The left boundary condition on w implies that,

$$\begin{aligned} w(0,t) &= u(0,t) - T(0) && (81)\\ &= g(t) - b && (82)\\ &= 0, && (83) \end{aligned}$$

while the right boundary condition on w implies that,

$$\begin{aligned} w(L,t) &= u(L,t) - T(L) && (84)\\ &= h(t) - aL - b && (85)\\ &= 0. && (86) \end{aligned}$$

From this we have that,

$$\begin{aligned} b &= g(t), && (87)\\ a &= \frac{h(t)-b}{L}. && (88) \end{aligned}$$

This gives the form for w as,

$$\begin{aligned} w(x,t) &= u(x,t) - \left(\frac{h(t)-b}{L}x + g(t)\right) && (89)\\ &= u(x,t) - \left(\frac{h(t)-g(t)}{L}x + g(t)\right) && (90)\\ &= u(x,t) - S(x,t), && (91) \end{aligned}$$

where $S(x,t) = \left(\dfrac{h(t)-g(t)}{L}x + g(t)\right)$. The initial condition on w is now,

$$w(x,0) = u(x,0) - S(x,0). \tag{92}$$

Finally, we have the transformations,

$$\begin{aligned} u_t &= w_t + S_t, && (93)\\ u_{xx} &= w_{xx} + S_{xx} && (94)\\ &= w_{xx}, && (95) \end{aligned}$$

[18] A similar transformation can be found for the wave equation with inhomogeneous boundary conditions. The moral here is that time-dependent boundary conditions can be transformed into externally driven (AKA Forced or inhomogeneous) PDE with standard boundary conditions.

where $S_{xx}(x,t) = \left(\frac{h(t)-g(t)}{L}x + g(t)\right)_{xx} = 0$. Thus, the PDE on u becomes,

$$\frac{\partial w}{\partial t} = c^2 \frac{\partial^2 w}{\partial x^2} - \frac{\partial S}{\partial t}, \tag{96}$$

which implies that the PDE on u with time-dependent boundary conditions transforms to a PDE on w with the inhomogeneous term S_t.

5. Inhomogeneous Wave Equation on a closed and bounded spatial domain of $\mathbb{R}^{1+1}$

Consider the non-homogeneous one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + F(x,t) \quad , \tag{97}$$

$$x \in (0,L)\,, \qquad t \in (0,\infty)\,, \qquad c^2 = \frac{T}{\rho}. \tag{98}$$

with boundary conditions and initial conditions,

$$u(0,t) = u(L,t) = 0, \tag{99}$$

$$u(x,0) = u_t(x,0) = 0. \tag{100}$$

Letting $F(x,t) = A\sin(\omega t)$ gives the following Fourier Series Representation of the forcing function F ,

$$F(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi x}{L}\right), \tag{101}$$

where

$$f_n(t) = \frac{2A}{n\pi}(1-(-1)^n)\sin(\omega t). \tag{102}$$

5.1. **Educated Fourier Series Guessing.** Based on the boundary conditions we assume a Fourier sine series solution. However, the time-dependence is unclear. So, assume that,

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) G_n(t), \tag{103}$$

where $G_n(t)$ represents the unknown dynamics of the n-th Fourier mode. Using this assumption and (101)-(102), show by direct substitution that (97) yields the ODE,

$$\ddot{G}_n + \left(\frac{cn\pi}{L}\right)^2 G_n = \frac{2A}{n\pi}(1-(-1)^n)\sin(\omega t). \tag{104}$$

We have that,

$$u_{tt} - c^2 u_{xx} - F(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\ddot{G}_n(t) + \left(\frac{cn\pi}{L}\right)^2 \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) G_n(t) - \sum_{n=1}^{\infty} f_n(t)\sin\left(\frac{n\pi x}{L}\right) \tag{105}$$

$$= \sum_{n=1}^{\infty}\left[\ddot{G}_n(t) + \left(\frac{cn\pi}{L}\right)^2 G_n(t) - f_n(t)\right]\sin\left(\frac{n\pi}{L}x\right) = 0. \tag{106}$$

Since $\sin\left(\frac{n\pi}{L}x\right) \neq 0$ for all x we require that,

$$\ddot{G}_n(t) + \left(\frac{cn\pi}{L}\right)^2 G_n(t) - f_n(t) = 0 \iff \tag{107}$$

$$\ddot{G}_n + \left(\frac{cn\pi}{L}\right)^2 G_n = \frac{2A}{n\pi}(1-(-1)^n)\sin(\omega t) \tag{108}$$

5.2. **Solving for the Dynamics.** The solution to (104) is given by,

$$G_n(t) = G_n^h(t) + G_n^p(t), \tag{109}$$

where $G_n^h(t) = B_n\cos\left(\frac{cn\pi}{L}t\right) + B_n^*\sin\left(\frac{cn\pi}{L}t\right)$ is the homogeneous solution and $G_n^p(t)$ is the particular solution to (104).

5.2.1. *Particular Solution - I.* If $\omega \neq cn\pi/L$ then what would the choice for $G_n^p(t)$ be, assuming you were solving for $G_n^p(t)$ using the method of undetermined coefficients? DO NOT SOLVE FOR THESE COEFFICIENTS

If the forcing function to (104) is given by $f_n(t) = \dfrac{2A}{n\pi}(1-(-1)^n)\sin(\omega t) = \alpha_n \sin(\omega t)$ then we should choose the particular solution $G_n(t) = \beta_n \cos(\omega t) + \beta_n^* \sin(\omega t)$ for $\omega \neq cn\pi/L$. This would then give the general solution to the inhomogeneous PDE,

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left[B_n \cos\left(\frac{cn\pi}{L}t\right) + B_n^* \sin\left(\frac{cn\pi}{L}t\right) + \beta_n \cos(\omega t) + \beta_n^* \sin(\omega t)\right], \tag{110}$$

where β_n and β_n^* are undetermined coefficients found by standard ODE techniques.

5.2.2. *Particular Solution - II.* If $\omega = cn\pi/L$ then what would the choice for $G_n^p(t)$ be, assuming you were solving for $G_n^p(t)$ using the method of undetermined coefficients? DO NOT SOLVE FOR THESE COEFFICIENTS

If the forcing function to (104) is given by $f_n(t) = \dfrac{2A}{n\pi}(1-(-1)^n)\sin(\omega t) = \alpha_n \sin(\omega t)$ and $\omega = cn\pi/L$ then we should choose the particular solution $G_n(t) = \beta_n t\cos(\omega t) + \beta_n^* t\sin(\omega t)$ for $\omega \neq cn\pi/L$. This would then give the general solution to the inhomogeneous PDE,

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left[B_n \cos\left(\frac{cn\pi}{L}t\right) + B_n^* \sin\left(\frac{cn\pi}{L}t\right) + \beta_n t\cos(\omega t) + \beta_n^* t\sin(\omega t)\right], \tag{111}$$

where β_n and β_n^* are undetermined coefficients found by standard ODE techniques.

5.2.3. *Physical Conclusions.* For the Particular Solution - II, what is $\lim_{t\to\infty} u(x,t)$ and what does this limit imply physically?

There is a t-prefactor on this particular solution, which causes the amplitudes of oscillations to grow as a function of time. Formally, this limit becomes unbounded as $t \to \infty$, which is the hallmark of resonance. The moral here is that systems that are prone to oscillate can be made to resonate under the right external force. This is seen in ODE's via mass-spring systems but is just as true for any ideal oscillatory systems.