MATH 332 - Linear Algebra
Homework 3, Summer 2009

June 23, 2009
Due: June 29, 2009

Linear Independence - Matrix Transformation - Matrix Operations

1. Given that,

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 6 \\ -7 & 5 & 3 \\ 9 & -3 & 3 \end{bmatrix},$$

and observe that the first column plus twice the second column equals the third column. Find a nontrivial solution of $\mathbf{Ax} = \mathbf{0}$.

Hint: Row reduction will find nontrivial solutions to the system. However, it is unnecessary to use row reduction.

2. Determine the values of h for which the vectors are linearly dependent.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}$$

3. Suppose the vectors $\mathbf{v_1}, \ldots, \mathbf{v_p}$ span $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Suppose, as well, that $T(\mathbf{v}_i) = \mathbf{0}$ for $i = 1, \ldots, p$. Show that T is the zero-transformation. That is, show that if $\mathbf{x}$ is any vector in $\mathbb{R}^n$, then $T(\mathbf{x}) = \mathbf{0}$.

Hint: If $\mathbf{v_1}, \ldots, \mathbf{v_p}$ generates $\mathbb{R}^n$ then how can **any** vector in $\mathbb{R}^n$ be written?

4. In quantum mechanics spin one-half particles, typically an electron[1] , can be characterized by the following vectors:

$$\mathbf{e}_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_d = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where $\mathbf{e}_u$ represents spin-up and $\mathbf{e}_d$ represents spin-down.[2] The following matrices:

$$\mathbf{S}_+ = \hbar \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{S}_- = \hbar \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

where $\hbar$ is Planck's constant, are linear transformations, which act on $\mathbf{e}_u$ and $\mathbf{e}_d$.

(a) Compute and describe the effect of the transformations, $\mathbf{S}_+(\mathbf{e}_u + \mathbf{e}_d)$, and $\mathbf{S}_-(\mathbf{e}_u + \mathbf{e}_d)$.

(b) $\mathbf{S}_+$ and $\mathbf{S}_-$ are projection transformations. Projection transformations are known to destroy information. Justify this in the case of $\mathbf{S}_+$ and $\mathbf{S}_-$ by showing that for any vector $\mathbf{b} \in \mathbb{R}^2$ there does NOT exist a **unique** $\mathbf{x} \in \mathbb{R}^2$, which satisfies $\mathbf{S}_+(\mathbf{x}) = \mathbf{b}$ or $\mathbf{S}_-(\mathbf{x}) = \mathbf{b}$.

[1] In general these particles are called fermions. http://en.wikipedia.org/wiki/Spin-1/2, http://en.wikipedia.org/wiki/Fermions

[2] In quantum mechanics, the concept of spin was originally considered to be the rotation of an elementary particle about its own axis and thus was nsidered analogous to classical angular momentum subject to quantum quantization. However, this analogue is only correct in the sense that spin obeys e same rules as quantized angular momentum. In 'reality' spin is an intrinsic property of elementary particles and it is the roll of quantum mechanics understand how to associate quantized particles with spin to their associated background field in such a way that certain field properties/symmetries e preserved. This is studied in so-called quantum field theory. http://www.physics.thetangentbundle.net/wiki/Quantum_mechanics/spin, http: en.wikipedia.org/wiki/Spin_(physics), http://en.wikipedia.org/wiki/Quantum_field_theory

5. We define the commutator and anti-commutation functions on matrices as[3],

$$[A, B] = AB - BA, \quad \{A, B\} = AB + BA. \tag{1}$$

The following matrices are the so-called Pauli spin matrices and have interesting commutation and anti-commutation relations and gives us fine setting to practice our matrix algebra. [4]

$$\sigma_1 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2}$$

Using the previous definitions show the following:

(a) $\sigma_i^2 = \mathbf{I}$ for $i = 1, 2, 3$. [5]

(b) $[\sigma_i, \sigma_j] = 2i \sum_{k=1}^{3} \epsilon_{ijk} \sigma_k$ for $i = 1, 2, 3$ and $j = 1, 2, 3$. [6]

(c) $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbf{I}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$. [7]

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[3] A commutator is a function, which takes in two matrices and returns one and is, in some sense, a measure of the binary operations lack of mmutativity.

[4] The Pauli spin matrices are a set of Hermitian matrices, which are *unitary*. They have found several uses including describing strong interaction mmetries in particle physics, logic gates in quantum information theory and representation of finite groups in abstract algebra.

[5] This statement encapsulates both the symmetric unitary properties of the matrices.

[6] Here we are using the so-called Levi-Civita symbol. This symbol is used to encode the following commonly encountered information,

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1) \text{ or } (3,1,2), \\ -1, & \text{if } (i,j,k) \text{ is } (3,2,1), (1,3,2) \text{ or } (2,1,3), \\ 0, & \text{if } i = j \text{ or } j = k \text{ or } k = i \end{cases} \tag{3}$$

[7] Here we use the so-called Kronecker delta function, which encodes the, also common, information,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j \end{cases} \tag{4}$$

5) One way to find the trivial solutions of $A\vec{x}=\vec{0}$ is to row reduce

$$\left[\begin{array}{ccc|c} 4 & 1 & 6 & 0 \\ -7 & 5 & 3 & 0 \\ 9 & -3 & 3 & 0 \end{array}\right]$$

However, if we note that

$$\vec{a}_3 = \vec{a}_1 + 2\vec{a}_2 \iff a_1 + 2a_2 - \vec{a}_3 = 0 \iff$$

$$\iff x_1\vec{a}_1 + x_2\vec{a}_2 - x_3\vec{a}_3 = 0 \quad \text{where } x_1=1,\ x_2=2,\ x_3=-1.$$

We have that since,

$$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = A\vec{x} = 0$$

(*) $(x_1, x_2, x_3) = (1, 2, -1)$ is a nontrivial solution to $A\vec{x}=\vec{0}$.

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Using Row reduction gives,

$$\left[\begin{array}{ccc|c} 4 & 1 & 6 & 0 \\ -7 & 5 & 3 & 0 \\ 9 & -3 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 4 & 1 & 6 & 0 \\ 0 & 27 & 54 & 0 \\ 0 & -21 & -42 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 4 & 1 & 6 & 0 \\ 0 & 27 & 54 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 4 & 1 & 6 & 0 \\ 0 & 1 & 54/27 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 4 & 0 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_3 = t \text{ (free)} \\ x_2 = -2t \\ x_1 = -t \end{array} \Rightarrow \vec{x} = \begin{bmatrix} -1 \\ -2 \\ +1 \end{bmatrix} t \text{ , } t \in \mathbb{R}$$

If $t = -1$ then,

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$ which is as before in (*)

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Homework 3 - Solutions

1. Using $\vec{v}_1, \vec{v}_2, \vec{v}_3$ construct the matrix,

$$V = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = \begin{bmatrix} 1 & -5 & 1 \\ -1 & 7 & 1 \\ -3 & 8 & h \end{bmatrix}$$

If the columns of V are linearly independent if and only if. $V\vec{x} = \vec{0}$ has only the trivial solution. Thus

$$\left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ -1 & 7 & 1 & 0 \\ -3 & 8 & h & 0 \end{array}\right] \underset{R2 = R1+R2}{\overset{R3=3R3+R1}{\sim}} \left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -7 & 3+h & 0 \end{array}\right] \overset{R3=2R3+7R2}{\sim}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2h+20 & 0 \end{array}\right]$$

Shows that $V\vec{x} = 0$ has nontrivial solutions for

$2h+20 = 0 \iff h = -10$

Hence, if $h = -10$ then $V\vec{x} = 0$ has nontrivial solutions and if $V\vec{x} = \vec{0}$ has nontrivial solutions then the columns of V are linearly dependent.

4) Proof:

If $\mathrm{span}\{v_1, \cdots, v_p\} = \mathbb{R}^n$ then for any $\vec{x} \in \mathbb{R}^n$ there exists $c_1, c_2, c_3, \cdots, c_p \in \mathbb{R}$ such that

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \cdots + c_p\vec{v}_p$$

Thus, for any $\vec{x} \in \mathbb{R}^n$

$T(x) = T(c_1\vec{v}_1 + \cdots + c_p\vec{v}_p) =$

(by linearity)

$= T(c_1\vec{v}_1) + T(c_2\vec{v}_2) + T(c_3\vec{v}_3) + \cdots + T(c_p\vec{v}_p) =$

(linearity again)

$= c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + c_3T(\vec{v}_3) + \cdots + c_pT(\vec{v}_p) =$

(by assumption)

$= c_1 \cdot 0 + c_2 \cdot 0 + c_3 \cdot 0 + \cdots + c_p \cdot 0 =$

$= 0$

Hence for any $\vec{x} \in \mathbb{R}^n$, $T(\vec{x}) = 0$ which implies T is the zero transformation.

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3) • $S_+(\vec{e}_u + \vec{e}_d) = S_+\vec{e}_u + S_+\vec{e}_d = \vec{S}_{+1} \cdot 1 + \vec{S}_{+2} \cdot 0 + \vec{S}_{+1} \cdot 0 + \vec{S}_{+2} \cdot 1 =$

$= \hbar \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \hbar \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \hbar \cdot \vec{0} + \hbar \vec{e}_u \quad (1)$

• $S_-(\vec{e}_u + \vec{e}_d) = 1 \cdot \vec{S}_{-1} + 0 \cdot \vec{S}_{-2} + 0 \cdot \vec{S}_{-1} + 1 \cdot \vec{S}_{-2} =$

$= \hbar \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \hbar \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \hbar \vec{e}_d + \hbar \cdot \vec{0} \quad (2)$

From (1),(2) we see that,

$S_+(\vec{e}_u) = \vec{0}$
$S_+(\vec{e}_d) = \vec{e}_u$
$S_-(\vec{e}_u) = \vec{e}_d$
$S_-(\vec{e}_d) = \vec{0}$

$\Rightarrow$

- S_+ transforms a down vector into an up vector
- S_+ transforms an up vector into a null vector (a nothing).
- S_- transforms an up vector into a down vector
- S_- transforms a down vector into a null vector.

b) The matrix representation of $S_+(\vec{x}) = \vec{b}$ is

$$\left[\begin{array}{cc|c} 0 & 1 & b_1 \\ 0 & 0 & b_2 \end{array}\right]$$

and $S_-(\vec{x}) = \vec{b}$,

$$\left[\begin{array}{cc|c} 0 & 0 & b_1 \\ 1 & 0 & b_2 \end{array}\right]$$

Each augmented matrix has a zero row which implies a free variable. Thus, if $S_+(\vec{x}) = \vec{b}$ or $S_-(\vec{x}) = \vec{b}$ have solutions they are not unique.

This implies that for some $\vec{b} \in \mathbb{R}^2$ there exists $x_1, x_2 \in \mathbb{R}^2$ such that

$$S_+(\vec{x}_1) = \vec{b} \quad \text{and} \quad S_+(\vec{x}_2) = \vec{b}.$$

Thus, there is no inverse transformation which will always get us back after we apply S_+ or S_-.

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Homework #1 Solution:

1.
Given

$$\sigma_1 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$
$$\sigma_3 = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

a)
$$\begin{aligned}
\sigma_1^2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\sigma_2^2 &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} -i^2 & 0 \\ 0 & -i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\sigma_3^2 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

b) Note:

- $[B, A] = BA - AB = -(AB - BA) = -[A, B]$
- $[\sigma_i, \sigma_i] = \sigma_i^2 - \sigma_i^2 = 0$

Thus we only need to consider, $(i, j) \in \{(1,2), (1,3), (2,3)\}$

$$\begin{aligned}
[\sigma_1, \sigma_2] &= \sigma_1\sigma_2 - \sigma_2\sigma_1 = \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \\
&= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix} = 2i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 2i\sigma_3
\end{aligned}$$

and

$$2i\sum_{k=1}^{3} \epsilon_{12k}\sigma_k = 2i\epsilon_{123}\sigma_3 = 2i\sigma_3$$

$$\begin{aligned}
[\sigma_1, \sigma_3] &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \\
&= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = -2i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -2i\sigma_2
\end{aligned}$$

and

$$2i\sum_{k=1}^{3} \epsilon_{13k}\sigma_k = 2i(\epsilon_{131}\sigma_1 + \epsilon_{132}\sigma_2 + \epsilon_{133}\sigma_3) = -2i\sigma_2$$

$$\begin{aligned}
[\sigma_2, \sigma_3] &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \\
&= \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} - \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = 2i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2i\sigma_1
\end{aligned}$$

and

$$\sum_{k=1}^{3} \epsilon_{23k}\sigma_k = 2i\epsilon_{131}\sigma_1 = 2i\sigma_1$$

$$\begin{aligned}
2i\sigma_3 &= [\sigma_1, \sigma_2] = -[\sigma_2, \sigma_1] = -2i\sum_{k=1}^{3} \epsilon_{21k}\sigma_k = \\
&= -2i\epsilon_{213}\sigma_3 = 2i\sigma_3
\end{aligned}$$

c) Similarly

$$\begin{aligned}
\{\sigma_i, \sigma_i\} &= \sigma_i^2 + \sigma_i^2 = 2\sigma_i^2 = 2I = 2I\delta_{ii}, \qquad i = 1, 2, \ldots \\
&\text{and} \\
\{\sigma_i, \sigma_j\} &= \{\sigma_j, \sigma_i\} \\
&\text{and} \\
\{\sigma_1, \sigma_2\} &= \sigma_1\sigma_2 + \sigma_2\sigma_1 = 0 \\
\{\sigma_1, \sigma_3\} &= \sigma_1\sigma_3 + \sigma_3\sigma_1 = 0 \\
\{\sigma_2, \sigma_3\} &= \sigma_2\sigma_3 + \sigma_3\sigma_2 = 0
\end{aligned}$$

2. We begin by writing the 3 linear equations (3),(4),(5) as the augmented matrix,

$$\left[\begin{array}{ccc|c} 6 & 18 & 1 & 20 \\ -1 & -3 & 8 & 4 \\ 5 & 15 & -9 & 11 \end{array}\right] \begin{array}{c} R1 \to R3 \\ R3 \to R2 \\ R2 \to R1 \end{array} \sim \left[\begin{array}{ccc|c} -1 & -3 & 8 & 4 \\ 5 & 15 & -9 & 11 \\ 6 & 18 & -4 & 20 \end{array}\right] \sim$$

$$\sim_{\substack{R2=5R1+R2 \\ R3=6R1+R3}} \left[\begin{array}{ccc|c} -1 & -3 & 8 & 4 \\ 0 & 0 & 40-9 & 11+20 \\ 0 & 0 & 44 & 44 \end{array}\right] \sim_{\substack{R3=R3/44 \\ R2=R2/44}}$$

$$\sim_{R1=-R1} \left[\begin{array}{ccc|c} 1 & 3 & -8 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim_{R3=R3+R2} \left[\begin{array}{ccc|c} 1 & 3 & 8 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\sim_{R1=R1+8R2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Which corresponds to the row equivalent linear system,

$$x_1 + 3x_2 = 4$$

$$x_3 = 1$$

Letting $x_2 = t$ implies that the general solution set is given by,

$$(\star) = \begin{array}{l} x_1 = -3t + 4 \\ x_2 = t \\ x_3 = 1 \end{array} \quad , \qquad t \in \Re$$