MATH348: LINEAR SYSTEMS OF EQUATIONS AND GAUSSIAN ELIMINATION

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Good evening. Do not attempt to adjust your radio, there is nothing wrong. We have taken control as to bring you this special show. We will return it to you as soon as you are grooving.

1. Linear Systems of Equations

In science it is common to arrive at systems of linear equations, which takes the form

$$
\begin{aligned}
a_{11}x_1 + \ a_{12}x_2 \ +a_{13}x_3 + \cdots + \ a_{1n}x_n &= \ b_1,\\
a_{21}x_1 + \ a_{22}x_2 \ +a_{23}x_2 + \cdots + \ a_{2n}x_n &= \ b_2,\\
a_{31}x_1 + \ a_{32}x_2 \ +a_{33}x_3 + \cdots + \ a_{3n}x_n &= \ b_3,\\
\vdots \qquad \vdots \qquad\qquad \vdots \qquad\quad & \quad \vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= \ b_m.
\end{aligned} \tag{1}
$$

Some examples include (the linear nature of the problem appears in bold):

1. Single **Line**: $a_{11}x_1 + a_{12}x_2 = b_1$ or $y = mx + b$ if $x_1 = y$, $m = -a_{12}/a_{11}$ and $b = b_1/a_{11}$.
2. Three **Lines**:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 &= b_1,\\
a_{21}x_1 + a_{22}x_2 &= b_2,\\
a_{31}x_1 + a_{32}x_2 &= b_3.
\end{aligned} \tag{2}
$$

3. A Single **Plane**: $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$ where $x_1 = x$, $x_2 = y$, $x_3 = z$.
4. Three **Planes**:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1,\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2,\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3.
\end{aligned} \tag{3}
$$

5. A first order **linear** ordinary differential equation: $\dfrac{dy}{dt} = a(t)y + b(t)$, where the right–hand side defines the linear system.
6. A second order **linear** ordinary differential equation: Here we consider the equation $ay'' + by' + cy = f(t)$ under the substitution $v = y'$ which gives two first order equations

$$v' = -\frac{b}{a}v - \frac{c}{a}y + \frac{f(t)}{a}, \tag{4}$$

$$y' = v, \tag{5}$$

where the right–hand side defines the linear system of equations.

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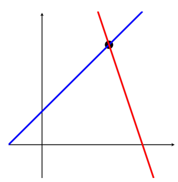
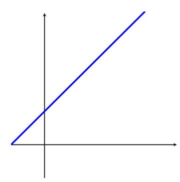
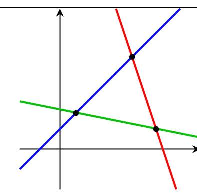
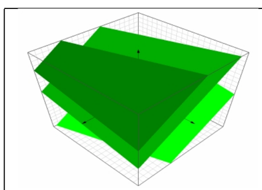
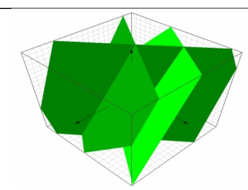
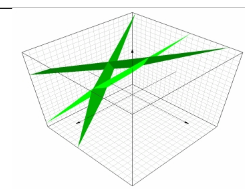
What is important to notice about all of these statements is that the is always a finite number of equations, $m<\infty$, and a finite number of dimensions, $n<\infty$.[1] Thinking about the first four examples above, the goal is to find the unknown values of x_i such that the equations are simultaneously true. Based on the simplest case where $m=n=1$, we can generally expect this to break into a tracheotomy of outcomes. That is if $a,b\in\mathbb{R}$ then the equation $ax=b$ has one of the following solution–sets:

1. The solution set has exactly one element. If $a\neq 0$ then $x=a^{-1}b$.
2. The solution set has infinitely–many elements. If $a=0$ and $b=0$ then $x\in\mathbb{R}$.
3. The solution set is empty. If $a=0$ and $b\neq 0$ then there is no x which solves the problem.

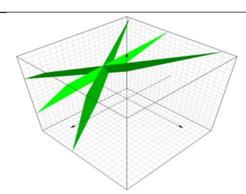
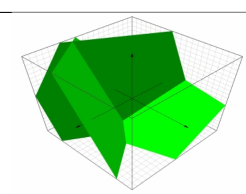
Geometrically, we can think about various arrangements of two–lines:

		
A solution exists and is unique	Solutions exist but are not unique (same line)	No solutions exist

You may also think about multiple planes in space:

		
Three parallel planes	Two parallel planes with one intersecting both	Non parallel non intersecting

Here we see there are many more ways planes can be oriented so that there is no solution to the linear system. Some examples of unique and non–unique solutions include are:

	
Three planes intersecting at a single point	Three planes intersecting at many points

The same sort of situations occur in higher dimensions of space but we cannot visualize the geometry. The point here is that we know the solution–set of a linear system should fall into one of these three cases. However, we need to quantify this quality.

[1] Those that have studied partial differential equations should notice that separation of variables led to infinitely many ordinary differential equations and so for PDE we loose this nicety.

2. ROW–REDUCTION AND SOLUTIONS TO LINEAR SYSTEMS

In order to solve a linear system we must define a procedure that we can apply to Eq. (1) that does not change the solution-set to Eq. (1) but at the same time makes this set obvious. First, what we should notice is that in our simple example where $m = n = 1$ the solution–set depends on a, b, not x. In terms of lines, we are saying that the points of intersection are determined by the orientations/"slopes", a_{ij}, and origin–offset/"y–intercepts", b_i, of the individual lines. In other words, the intersection of linear/flat objects in space depends on the orientation and position of these objects; pretty sensible really. With this in mind, we rewrite the original system of linear equations in the following augmented matrix form,

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{33} & \cdots & a_{mn} & b_m \end{array}\right] \tag{6}$$

We should note that each row of the augmented matrix represents a single equation. Our goal is to manipulate this representation to the point where the solution–set is obvious. Rules of manipulating an augmented matrix are often called row–operations and are just restatements of the rules of algebra applied to the original system. Row–operations that do not change the solution-set of the linear system represented by the augmented matrix are:

1. Row Interchange: Any two rows in an augmented matrix can be swapped.
2. Row Scaling: Any row in an augmented matrix can be multiplied by a non–zero scalar.
3. Row Replacement: Any row in an augmented matrix can be replaced by the original row summed with a multiple of another row.

The application of these rules to an augmented matrix, so as to make obvious the solution–set to the linear system, is often called the row–reduction algorithm or Gaussian elimination. While this can be described in words, it can be explained visually in a very straight–forward way. In the following we use row–operations to solve the linear system,

$$x_2 + 2x_3 = 3 \tag{7}$$

$$4x_1 + 5x_2 + 6x_3 = 7 \tag{8}$$

$$8x_1 + 9x_2 + 10x_3 = 11, \tag{9}$$

which represents three planes in space. Application of the row–reduction algorithm to this system's augmented matrix gives,

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{ccc|c} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \end{array}\right] \tag{10}$$

$$= \left[\begin{array}{ccc|c} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \end{array}\right] \quad \boxed{R2 \to R1} \quad \boxed{R3 \to R2} \quad \boxed{R1 \to R3} \tag{11}$$

$$\sim \left[\begin{array}{ccc|c} 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 0 & 1 & 2 & 3 \end{array}\right] \tag{12}$$

$$= \left[\begin{array}{ccc|c} \textcircled{4} & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 0 & 1 & 2 & 3 \end{array}\right] \quad \boxed{\times -2} \; + \tag{13}$$

$$\sim \left[\begin{array}{ccc|c} \textcircled{4} & 5 & 6 & 7 \\ 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 \end{array}\right] \tag{14}$$

$$= \left[\begin{array}{ccc|c} 4 & 5 & 6 & 7 \\ 0 & \textcircled{-1} & -2 & -3 \\ 0 & 1 & 2 & 3 \end{array}\right] \quad \boxed{\times 1} \; + \tag{15}$$

$$\sim \left[\begin{array}{ccc|c} 4 & 5 & 6 & 7 \\ 0 & \textcircled{-1} & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad + \; \boxed{\times 5} \tag{16}$$

$$\sim \left[\begin{array}{ccc|c} 4 & 0 & -4 & -8 \\ 0 & \textcircled{-1} & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \boxed{\times \frac{1}{4}} \quad \boxed{\times -1} \tag{17}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]. \tag{18}$$

The linear system that corresponds the final augmented matrix is,

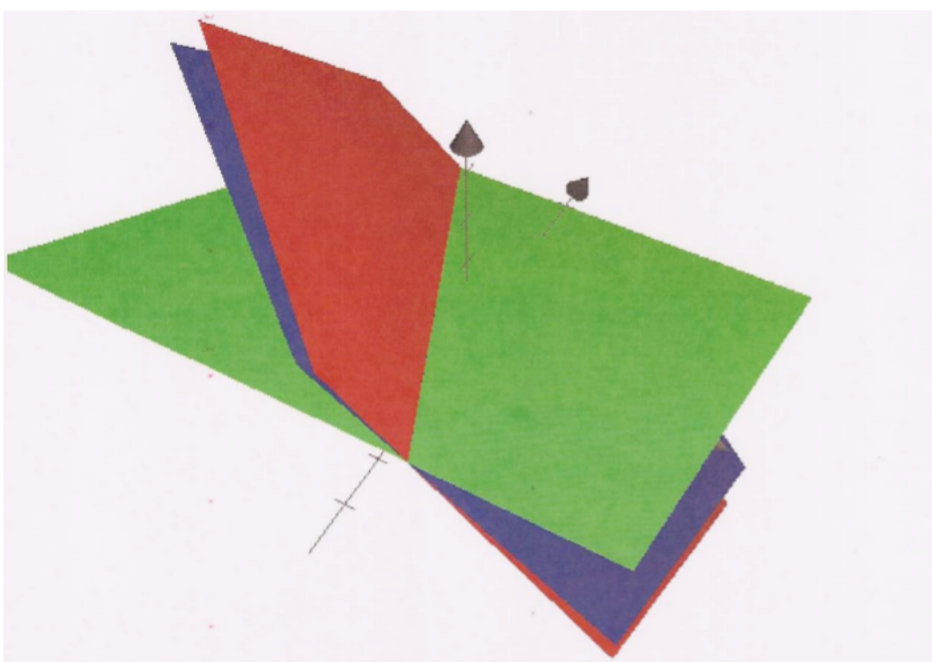
$$x_1 - x_3 = -2 \tag{19}$$

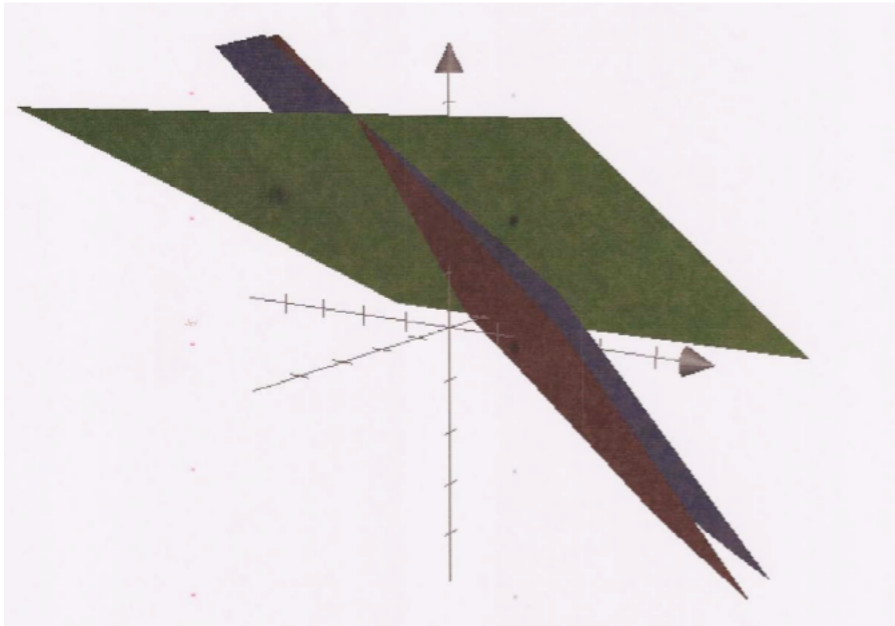
$$x_2 + 2x_3 = 3. \tag{20}$$

The solution to this system is given by,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 - 2 \\ 3 - 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \quad x_3 \in \mathbb{R}. \tag{21}$$

The important point here is that since row–operations do not change the solution to a linear system, this solution is the same as the solution to our original linear system. The solution is made obvious by the use of row–reduction which decouples variables from equations. Notice that in the first system, the variable x_2 appears in all equations, while in the final system it appears only in the second equation. The decoupling of variables from equations is the goal of the row–reduction algorithm. The next two pictures are visualizations of these there planes, which intersect at a line parameterized by the solution $\mathbf{x}$ given above.





3. PROBLEMS

The following problems will give you practice with row–reduction of linear systems. For each you should encode the system in an augmented matrix and apply row–reduction to find the general solution to the linear system. When possible you should discuss the geometric meaning of the solution set.

1. Solve the linear system,

$$x_1 - 3x_2 = 5 \tag{22}$$
$$-x_1 + x_2 + 5x_3 = 2 \tag{23}$$
$$x_2 + x_3 = 0 \tag{24}$$

2. Solve the linear system,

$$6x_1 + 18x_2 - 4x_3 = 20 \tag{25}$$
$$-x_1 - 3x_2 + 8x_3 = 4 \tag{26}$$
$$5x_1 + 15x_2 - 9x_3 = 11 \tag{27}$$

3. Solve the linear system,

$$x_1 + 2x_2 + x_3 = 4 \tag{28}$$
$$x_2 - x_3 = 1 \tag{29}$$
$$x_1 + 3x_3 = 0 \tag{30}$$

4. Solve the linear system,

$$x_1 + 2x_2 + 3x_3 = 10 \tag{31}$$
$$2x_1 + 4x_2 + 6x_3 = 20 \tag{32}$$
$$3x_1 + 6x_2 + 9x_3 = 30 \tag{33}$$

5. Solve the linear system,

$$5x_1 + 3x_2 = 22 \tag{34}$$

$$-4x_1 + 7x_2 = 20 \tag{35}$$

$$9x_1 - 2x_2 = 15 \tag{36}$$

6. Solve the linear system,

$$-8x_1 - 2x_2 - 9x_3 = 2 \tag{37}$$

$$6x_1 + 4x_2 + 8x_3 = 1 \tag{38}$$

$$4x_1 + 4x_3 = -2 \tag{39}$$

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