MATH 332 - Linear Algebra July 16, 2009

Homework 6, Summer 2009 **Due**: July 20, 2009

Vector Spaces - Subspaces - Bases and Coordiantes - Classical Matrix Spaces - Abstract Vector Spaces

1. (a) Verify that the set of all n-times continuously differentiable functions on $[a,b]$, which satisfies the homogeneous linear ordinary differential equation $L[y]=0$,

$$V=\left\{y\in C^{(n)}[a,b] : L[y]=a_n(t)\frac{d^ny}{dt^n}+a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}}+\cdots+a_0(t)y=0,\ \text{where } a_0,\ldots,a_n\in C[a,b]\right\},$$

is a vector space under addition of functions and scalar multiplication.

(b) Prove that if H is the set of all polynomials up to degree n, such that $p(0)=0$, then H is a subspace of $\mathbb{P}_n$.

(c) Prove that if $H=\{f\in C[a,b] : f(a)=f(b)\}$, then H is a subspace of $C[a,b]$.

2. The standard basis for $\mathbb{R}^2$ are the column vectors, $\{\mathbf{e}_1,\mathbf{e}_2\}$ of $\mathbf{I}_{2\times 2}$. In class we looked at the basis $\mathfrak{B}=\{[1,1]^{\mathrm{T}},[-1,1]^{\mathrm{T}}\}$. This basis is rotated $\frac{\pi}{4}$ radians counter-clockwise from the standard basis and does not preserve the notion of length from the standard coordinate system.

(a) Determine a basis for $\mathbb{R}^2$, which is rotated $\frac{\pi}{4}$ radians counter-clockwise from the standard basis and preserves the unit length associated with the standard basis.

(b) Show that, for this basis, the change-of-coordinates matrix $\mathbf{P}_{\mathfrak{B}}$ is such that, $\mathbf{P}_{\mathfrak{B}}\mathbf{P}_{\mathfrak{B}}^{\mathrm{T}}=\mathbf{P}_{\mathfrak{B}}^{\mathrm{T}}\mathbf{P}_{\mathfrak{B}}=\mathbf{I}_{2\times 2}$.

(c) Given that $[\mathbf{x}_1]_{\mathfrak{B}}=[\sqrt{2},\sqrt{2}]^{\mathrm{T}}$ determine $\mathbf{x}_1$ and given that $\mathbf{x}_2=\left[\frac{3}{\sqrt{2}},\frac{3}{\sqrt{2}}\right]^{\mathrm{T}}$ determine $[\mathbf{x}_2]_{\mathfrak{B}}$. Calculate the magnitude of both of the vectors previously calculated.

3. Given,

$$\mathbf{A}=\begin{bmatrix}-8&-2&-9\\6&4&8\\4&0&4\end{bmatrix},\quad \mathbf{w}=\begin{bmatrix}2\\1\\-2\end{bmatrix}.$$

(a) Is $\mathbf{w}$ in the column space of $\mathbf{A}$? That is, does $\mathbf{w}\in\text{Col }\mathbf{A}$?

(b) Is $\mathbf{w}$ in the null space of $\mathbf{A}$? That is, does $\mathbf{w}\in\text{Nul }\mathbf{A}$?

4. Given,

$$\mathbf{A}=\begin{bmatrix}2&-3&6&2&5\\-2&3&-3&-3&-4\\4&-6&9&5&9\\-2&3&3&-4&1\end{bmatrix}.$$

(a) Determine a basis and the dimension of Nul $\mathbf{A}$.

(b) Determine a basis and the dimension of Col $\mathbf{A}$.

(c) Determine a basis and the dimension of Row $\mathbf{A}$.

5. The Hermite polynomials are a sequence of orthogonal polynomials, which arise in probability, combinatorics and physics.[1] The first four polynomials in this sequence are given as,

$$H_0(x)=1,\quad H_1(x)=2x,\quad H_2(x)=-2+4x^2,\quad H_3(x)=-12x+8x^3,\quad x\in(-\infty,\infty).$$

(a) Show that $\mathfrak{B}=\{1,2x,-2+4x^2,-12x+8x^3\}$ is a basis for $\mathbb{P}_3$.
Hint: Determine the coordinate vectors of the Hermite polynomials relative to the standard basis.

(b) Let $\mathbf{p}(x)=7-12x-8x^2+12x^3$. Find the coordinate vector of $\mathbf{p}$ relative to $\mathfrak{B}$.
Hint: Determine $\{c_0,c_1,c_2,c_3\}$ such that $\mathbf{p}(x)=\sum_{i=0}^{3}c_iH_i(\text{x})$.

[1] In physics these polynomials manifest as the spatial solutions to Schrödinger's wave equation under a harmonic potential, which evolves the probability distribution of a quantum mechanical particle near an energy minimum. As it turns out there are infinitely-many Hermite polynomials and consequently one can show that this particle has infinitely-many allowed quantized energy levels, which are evenly spaced.