MATH 235 - Differential Equations w/ Honors April 13, 2011
Take-Home Exam I, Spring 2011 **Due**: April 25, 2011

The following problem involves mathematically modeling a bead on a rotating hoop in the singular limit of strong dampening, readings in fluid mechanics, power series solution to ODEs and will constitute your first examination.

This exam is open books, notes and internet. Though you may contact me, you are not to discuss your solutions with other students from this, or any, class. All CSM students are bound by the Student Honor Code. Any suspected violations of this code will be reported to the Student Judicial Panel for litigation.

Any corrections or clarifications to this exam will disseminated through the course blog.

Name: ______________________________

Student ID: ______________________________

Signature: ______________________________

By signing this document I imply the originality of my following work in compliance with the Colorado School of Mines Student Honor Code.

Overdamped Bead on a Rotating Hoop[1]

Description of the Physical Problem

Consider a bead of mass m, which is allowed to slide along a ridged hoop of radius r. This hoop is constrained to rotate at a constant angular velocity ω about its vertical axis. Pictorially, we have,

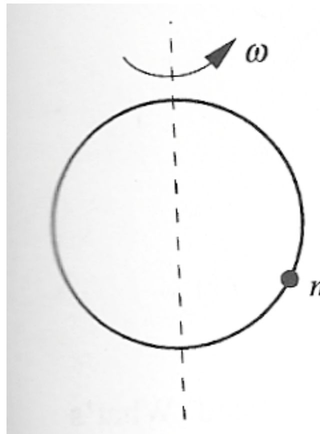

Figure 3.5.1

and assume that the bead is subjected to the forces:

- The gravitational force acting on the bead.
- Centrifugal force. [2]
- Frictional forces. [3]

We would like to find a differential equation that models the position of the bead as a function of the angular velocity ω. Before we do this we will need to instantiate some sort of coordinate system.

Derivation of Model Equation from Force Laws

First we define the angle, ϕ, to be the angle between the bead and the downward vertical direction. Clearly, we can restrict ϕ to the principle domain $-\pi < \phi \leq \pi$. Moreover, in terms of this angle we define the coordinate system,

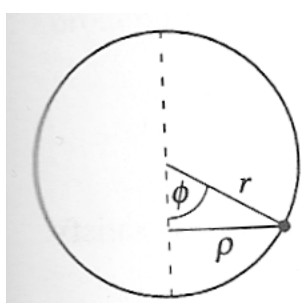

Figure 3.5.2

where we define ρ to be the distance of the bead from the vertical axis.

[1]This problem originally appears in Nonlinear Dynamics and Chaos, Strogatz, Perseus Books Publishing, 1994, pg61

[2]It should be noted that this *centrifugal force* is a so-called fictitious force and arises when treating rotating frames of reference. See `http://en.wikipedia.org/wiki/Centrifugal_force` for more information.

[3]You may want to imagine that the system is immersed in a vat of very viscous fluid so that the viscous damping opposes the motion of the bead.

A free body diagram of the bead is given by,

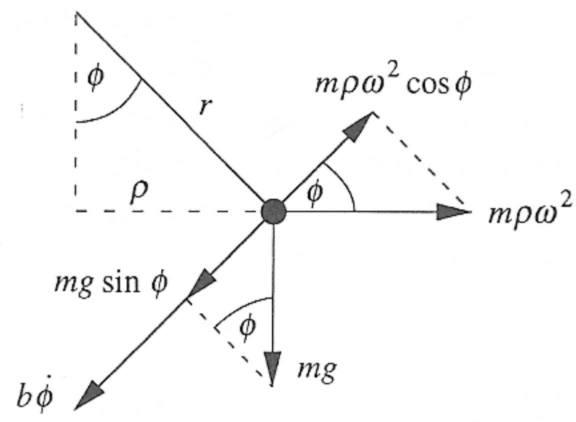

Figure 3.5.3

where,

- mg is a downward gravitation force
- $b\dot{\phi}$ is a tangential damping force
- $m\rho\omega^2$ is a sideways centrifugal force

Derivation of Model Equation from Force Laws - Questions

1. Determine ρ as a function of ϕ.
2. Substitute the ρ into the centrifugal term and noting that the tangential acceleration is $r\ddot{\phi}$ obtain the governing equation [4] ,
$$mr\ddot{\phi} = -b\dot{\phi} - mg\sin(\phi) + mr\omega^2 \sin(\phi)\cos(\phi). \tag{1}$$
3. What is the order of (1)? Is (1) linear?

So far, we haven't yet studied methods for second-order nonlinear equations. Our goal is to replace (1) with a simpler first-order equation, which we can analyze. In general, this is tricky businesses as it is not always appropriate to replace a second-order equation with a first-order equation. For now we will just assume that there is a regime for which the first-order equation is a correct model and apply our machinery to it. Later we will determine the limits which define the regime for which our simpler model is appropriate.

Analysis of the First-Order System

Assuming that the contribution of the term associated with the second-order derivative to the motion of the bead is negligible compared to the rest of the differential equation we arrive at the first-order equation,

$$b\dot{\phi} = f_\omega(\phi) = -mg\sin(\phi) + mr\omega^2 \sin(\phi)\cos(\phi). \tag{2}$$

1. The fixed points of (2) correspond to equilibrium positions for the bead. Determine all fixed points of (2).
2. Determine the bifurcation values of ω for (2).

[4] Here it is convenient to use the Newton's notation for derivatives, $\dot{y} = \dfrac{dy}{dt}$.

3. Interpret the physical relevance of these bifurcation values.

4. Using linearization classify the stability of the equilibrium points for values of ω at the bifurcation value and just above and below the bifurcation value.

5. Create a bifurcation diagram for this first-order equation and classify the bifurcation type.

Using our current tool-kit we can gather a comprehensive picture of the behaviors of the bead on the rotating hoop in terms of the parameter ω. However, we must keep in mind that (2) is an approximation to (1) and as with any approximation we must now determine sufficient criteria, which allows us to say that (2) well approximates (1).

Dimensional Analysis and Scaling

We now take the time to address the question, given (1) when is it appropriate to neglect the $mr\ddot{\phi}$ term and replace (1) with (2)? Well, it is clear that when $r \to 0$ or when $m \to 0$ we have that $mr\ddot{\phi} \to 0$ but these conditions physically simplify the problem to the point of triviality. In general finding sufficient conditions such that the approximation of (1) by (2) is complicated. The reason why is that currently (1) has the dimensions of force and it is unclear whether certain forces can be considered small with respect to others. We need to find a way to rewrite (1) so that we can compare the relative magnitudes of the terms in the equation. To do this we will try to rewrite (1) in a dimensionless form so that a term being small corresponds to a number being much less than unity.

To simplify the analysis of (1) we rewrite the equation in dimensionless form [5] with the substitution of,

$$\tau = \frac{t}{T}, \tag{3}$$

where T is a characteristic time scale to be chosen later and τ is regarded as a dimensionless time variable.

1. Using the previous substitution calculate $\ddot{\phi}$ and $\dot{\phi}$ in terms of the variable τ. [6]

2. Substitute the previous results into (1) to find the corresponding dimensionless equation,[7]

$$\frac{r}{gT^2}\frac{d^2\phi}{d\tau^2} = -\frac{b}{mgT}\frac{d\phi}{d\tau} + \sin(\phi)\left(\frac{r\omega^2}{g}\cos(\phi) - 1\right). \tag{4}$$

   In (4) all of the coefficients have the same units [8] and thus we can compare them relative to one another.

3. Show that setting the coefficient in front of the $\frac{d\phi}{d\tau}$ term to negative unity defines the characteristic time scale to be $T = \frac{b}{mg}$.

4. Using this characteristic time scale show that if, $b^2 \gg m^2 gr$, then the coefficient to $\frac{d^2\phi}{d\tau^2}$ is much less than unity. [9]

5. What is the physical interpretation of this regime? That is, what does, $\frac{r}{g}\left(\frac{mg}{b}\right)^2 \ll 1$, imply about b if all other parameters are fixed? What does it imply about the mass m if all other parameters are fixed?

[5] See also Lin and Segel (1998)

[6] To do this you will need to recall the chain-rule, which states $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$.

[7] Notice that the bifurcation relation, $\frac{r\omega^2}{g} = 1$, appears quite clearly in this dimensionless form as the pre-factor to the $\cos(\phi)$ term.

[8] Compare this to (1) where mr has units of MASS·LENGTH and mg has units of MASS·LENGTH·TIME^{-2}.

[9] This is the regime for which the approximation of (1) by (2) is valid.

A Paradox and Singular Perturbation Theory

We have determined the regime for which our approximation in valid. However, there is something fundamentally incorrect about this replacement. That is, in general a second-order ODE requires two initial conditions to specify unique solutions[10] Our approximation is a first-order ODE and thus only requires information concerning the initial position and will not, in general, satisfy both initial conditions! So, what, if anything, can we conclude from our approximation?

Consider the following phase-space plot, which is based on the following system of differential equations,

$$\Omega = \phi' \tag{5}$$

$$\epsilon\Omega' = f(\phi) - \Omega, \tag{6}$$

where $\epsilon = \dfrac{m^2 gr}{b^2}$.

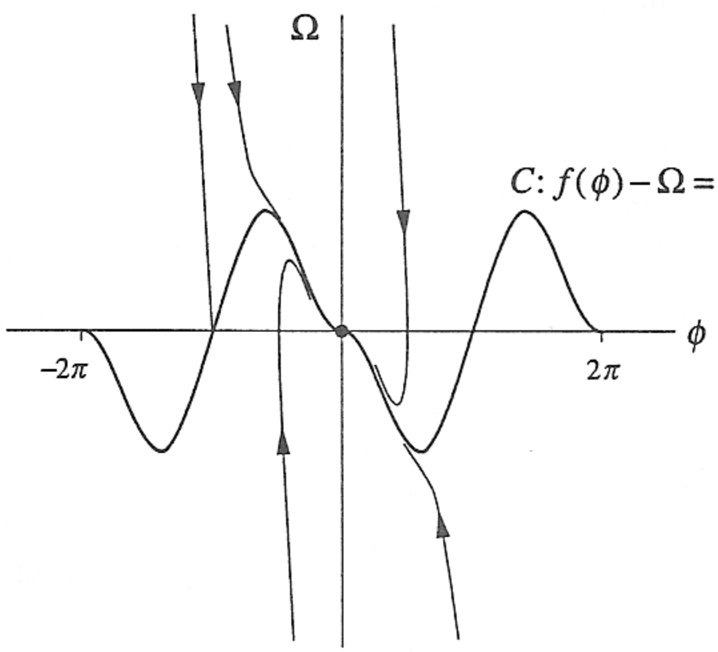

Figure 3.5.8

This graph implies that in the limit where $\epsilon \to 0$ all solutions tend to the origin as $t \to \infty$.[11] Solutions initially off the curve defined by $C : \Omega = f(\phi)$ tend to the curve before ultimately converging to the origin along f. Moreover, we have that for $\epsilon \to 0$, $\Omega' \to \infty$. This implies that, in this singular limit, solutions off the curve have large rates of change and make their way to the curve C quickly. That is, typical solutions have a rapid transient part, which corresponds to quick convergence to the curve C and a second part that describes a slow the evolution along the curve C.

This sort of analysis comes up frequently since in some limit of interest the term containing the highest order derivative vanishes. In this case the solution to the singular equation cannot, in general, be satisfied by the solution to the limiting equation.[12] The branch of mathematics called *singular perturbation theory* studies equations of just this type![13]

[10]These two initial conditions are generally taken to be the initial position and velocity.

[11]Notice that this limit occurs when $b \to \infty$.

[12]In our example the singular equation (1) cannot, in general, be satisfied by the solution to (2). However, this does not mean our solution is meaningless. It, in fact, has a great deal of meaning and is correct in some sort of limit.

[13]Jordan and Smith (1987) and Lin and Segel (1988)

A great example of this study of limiting cases comes from fluid flow modeling, which is considered to be an important open question in mathematics.[14] To gather some comparative insight into this field read the following websites,[15]

- http://en.wikipedia.org/wiki/Reynolds_number
- http://en.wikipedia.org/wiki/Laminar_flow
- http://en.wikipedia.org/wiki/Turbulent_flow
- http://en.wikipedia.org/wiki/Navier-Stokes_equations

and respond to the following questions:

1. What characterizes a laminar flow?[16]
2. For what limit of the Reynold's number is there a laminar flow associated with the dynamics of a fluid?
3. What characterizes the dynamics of the flow in the reciprocal limit?
4. What is an example of a physical system which could simultaneously produce fluids flows in both regimes?

Lastly, to give an application to the previous fluid concepts, read the attached article from Nature magazine[17] written by Hassan Aref titled *Order in chaos*, which concerns complicated spatial behaviors associated with slow fluid flows and their application to non-turbulent mixing. After reading this article:

1. Summarize the article in one to three paragraphs.
2. List five vocabulary words that you either didn't understand or hadn't encountered before the article.
3. List three questions questions you had after reading the article.

[14] See http://en.wikipedia.org/wiki/Millennium_Prize_Problems#Navier-Stokes_existence_and_smoothness and http://www.claymath.org/millennium/Navier-Stokes_Equations/ for more information.

[15] By read I mean that you should read, in detail, the introductory paragraphs and skim the rest of the article stopping/slowing for what you find interesting.

[16] In our case of the singular limit of $\epsilon \to 0$ we saw that the rapid transient played the role of a boundary layer - it is a thin layer of *time* that occurs near the boundary of $t = 0$.

[17] v401.p756-758

Power Series - Legendre Polynomials - Applications

Consider the problem of determining the potential due to a point charge, placed at the origin, at some position in space (x, y, z). The function representing the potential field satisfies Laplace's equation,

$$\triangledown^2 u = \Delta u = 0. \tag{7}$$

In Cartesian coordinates $\Delta u = u_{xx} + u_{yy} + u_{zz}$. However, due to natural symmetries of the physical problem it is often more convenient to express the problem in spherical coordinates. In that case $\Delta u = u_{\rho\rho} + 2\rho^{-1}u_\rho \rho^{-2}u_{\phi\phi} + \cot(\phi)r^{-2}u_\phi + r^{-2}\sin^{-2}(\phi)u_{\theta\theta}$, where the change of variables is given by $x = \rho\cos(\theta)\sin(\phi)$, $y = \rho\sin(\theta)\sin(\phi)$, $z = \rho\cos(\phi)$.

For an electrostatic potential it is common to assume symmetry with respect to θ, in this case one technique used to solve (7) leads to the ordinary differential equations with variable coefficients,

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \nu(\nu+1)y = 0, \;\; \nu \in \mathbb{C}. \tag{8}$$

This equation, (8), is called Legendre's equation and is commonly encountered in the study of potential fields in spherical coordinates. The solution to (8) can be found by power series techniques and leads to the orthogonal Legendre Polynomials.[18]

1. Assume that $y(x)$ has a power-series representation about $x = 0$ and solve (8).

   To do this assume that,

   $$y(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{9}$$

   and derive formula for y', y''. After this substitute the series representations into (8) to find,

   $$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty}\nu(\nu+1)a_n x^n = 0, \tag{10}$$

   which yields the relations,

   $$\begin{aligned} x^0 &: \;\; 2\cdot 1 a_2 + \nu(\nu+1)a_0 = 0 & (11)\\ x^1 &: \;\; 3\cdot 2a_3 + \left[-2+\nu(\nu+1)\right]a_1 = 0 & (12)\\ x^k &: \;\; (n+2)(n+1)a_{n+2} + \left[-n(n-1) - 2n + \nu(\nu+1)\right]a_n = 0, \;\; k = 2,3,4,5,\ldots & (13)\end{aligned}$$

2. Now assume that $\nu \in \mathbb{N}^+$ and using the initial conditions $(y(0), y'(0)) = (1, 0)$ or $(y(0), y'(0)) = (0, 1)$, show that there are polynomial solutions $P_\nu(x)$ of Legendre's equation.
   To produce polynomial solutions if ν is a positive integer, consider two cases:

   (a) If ν is even, use the initial condition $(y(0), y'(0)) = (1, 0)$. Then $a_1 = 0$ (why?) and all the coefficients of odd powers are zero. Moreover, the formula that relates a_{n+2} to a_n implies that $a_{\nu+2} = 0$. Therefore, all of the coefficients of higher even powers are also zero, and the solution is a polynomial (with no odd powers of x).

   (b) If ν is odd, use the initial condition $(y(0), y'(0)) = (0, 1)$ and obtain a polynomial solution with no even powers of x.

3. Find a formula for expressing the odd and even Legendre polynomials.[19]

[18] Adrien-Marie Legendre, 1752 - 1833, was French mathematician who is commemorated on the Eiffel Tower along with 71 other French scientists. Some career highlights include a proof that π is an irrational number and advancements with the elliptic integrals associated with Celestial mechanics.

[19] Dealing with the recurrence relation is not terrible but also not easy. It is best to use the factorization $\nu^2 + \nu - n^2 - n = [\nu + (n+1)](\nu - n)$. You may also find `http://mathworld.wolfram.com/LegendreDifferentialEquation.html` useful.

4. Compute $P_\nu(x)$ for $\nu = 1, 2, 3, 4, 5$.
5. Justify that $\alpha P_\nu(x),\ \ \alpha \in \mathbb{R}$ is a solution of (8). [20]
6. Read the attached material associated with the steady-state temperature in a ball and explain how this physical model is related to Legendre's equation. [21]

[20] This shows that the Legendre Polynomials are determined up to a multiplicative constant. Different fields in science choose the constant differently depending on the application.

[21] This material is from Differential Equations: Theory, Technique, and Practice, Simmons, Krants, 2007