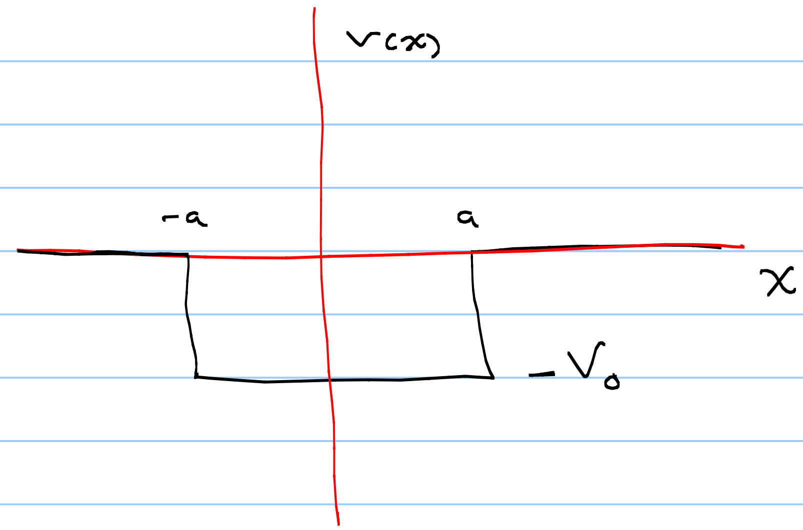
2_15_08

Note Title 2/14/2008

The finite square well



First: Bound states $E<0$

For $x < -a$ and $x > a$ $V(x) = 0$
we have already solve this and know that

$$\boxed{\begin{aligned} \psi(x) &= B e^{\kappa x} & x &< -a \\ \psi(x) &= F e^{-\kappa x} & x &> a \end{aligned}}$$

$$-\frac{\hbar^2}{2m}\psi'' = E\psi \Rightarrow \psi'' = -\frac{2mE}{\hbar^2}\psi = \kappa^2 \psi$$

Next for $x \in [-a, a]$

$$\kappa = \frac{\sqrt{-2mE}}{\hbar}$$

$$-\frac{\hbar^2}{2m}\psi'' - V_0\psi = E\psi$$

$$\psi'' = -\frac{2m}{\hbar^2}(V_0+E)\,\psi$$

$$\equiv -\ell^2 \psi \qquad \ell = \frac{\sqrt{2m(V_0+E)}}{\hbar}$$

we know that E must be greater than V_{min} [see p. 30 in Griffiths] Hence

So ℓ is real and positive

$$\psi'' + \ell^2 \psi = 0$$

$$\psi(x) = C\sin(\ell x) + D\cos(\ell x) \qquad -a < x < a$$

Since $V(x)$ is even, $\psi(x)$ must be even or odd. The reason is that if $\psi(x)$ is a solution of TISE then so is $\psi(-x)$

$$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x)$$

$$x \to -x \qquad \frac{d^2}{dx^2} \to \frac{d^2}{dx^2}$$

$$-\frac{\hbar^2}{2m}\psi''(-x) + \underset{V(-x)}{\underset{\parallel}{V(x)}}\,\psi(-x) = E\psi(-x)$$

QED

Then $\psi(x) \pm \psi(-x)$ is also a solution.

$\psi(x) + \psi(-x)$ is even
$\psi(x) - \psi(-x)$ is odd.

So let's take $\psi(x)$ to be even.

$$\psi(x) = \begin{cases} F e^{-\kappa x} & x > 0 \\ D\cos(\ell x) & 0 < x < a \\ \underbrace{\psi(-x)}_{\text{even}} & x < 0 \end{cases}$$

Continuity of ψ:

at $x = a$ $\quad F e^{-\kappa a} = D\cos(\ell a)$

Continuity of ψ'

at $x = a$ $\quad -\kappa F e^{-\kappa a} = -\ell D \sin(\ell a)$

$\Rightarrow \quad D\cos(\ell a) = \dfrac{\ell}{\kappa} D\sin(\ell a)$

$\Rightarrow \quad \kappa = \ell \tan(\ell a)$

$$\boxed{\ell = \frac{\sqrt{2m(V_0+E)}}{\hbar} \qquad \kappa = \frac{\sqrt{-2mE}}{\hbar}}$$

Since κ + ℓ depend on E this gives an equation to solve for the energy levels.

Simplify:

$$\kappa a = z \tan z \qquad \text{where} \quad \ell a = z$$

notice that $\ell^2 + \kappa^2 = \dfrac{2mV_0}{\hbar^2}$

so $$(\kappa a)^2 = \underbrace{\frac{a^2\, 2mV_0}{\hbar^2}}_{z_0^2} - \underbrace{(\ell a)^2}_{z^2}$$

$$z = \ell a$$
$$z_0 = \frac{a}{\hbar}\sqrt{2mV_0}$$

so $$\kappa a = \sqrt{z_0^2 - z^2}$$

$$\Rightarrow \quad \sqrt{z_0^2 - z^2} = z \tan z$$

$$\Rightarrow \quad \boxed{\tan z = \sqrt{(z_0/z)^2 - 1}}$$

z_0 is the "size" of the well

No Analytic Solution

<u>NB</u> $$\frac{z_0^2}{z^2} = \frac{a/\hbar\; 2mV_0}{a/\hbar\; 2m(E+V_0)} = \frac{V_0}{E+V_0}$$

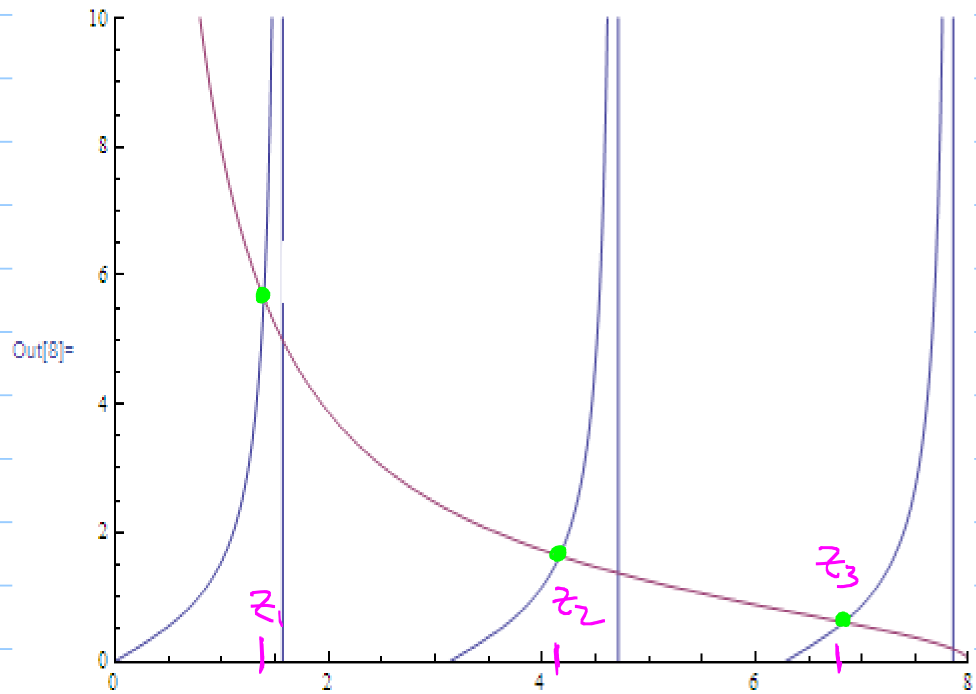
Remember $E > V_{min}$

<u>careful</u> by Griffiths defs $E < 0$
$V_0 > 0$ since the pot. is $-V_0$.

So although $\frac{z_0^2}{z^2} = \frac{V_0}{E+V_0}$ looks like it ought to be < 1, its actually greater that 1. Eg.

$$E = -3 \qquad V_0 = 5 \qquad \text{so}$$

$$\frac{5}{-3+5} > 1$$



mma code

$z_0 = 8$

$f[z\_] = \tan[z]$

$g[z\_] = \sqrt{z_0^2/z^2 - 1}$

starting point

FindRoot[f[z] == g[z], {z, 1}] = 1.39547 z_1

{z, 4} = 4.16483 z_2

{z, 7} 6.83067 z_3

$$\frac{z_0^2}{z^2} = \frac{V_0}{E + V_0}$$

$$(E + V_0)\frac{z_0^2}{z^2} = V_0$$

$$E = V_0\left(1 - z_0^2/z^2\right)$$

for z_1 $\quad E = V_0\left[1 - \left(\frac{8}{1.395}\right)^2\right]$

z_2 $\quad E = V_0\left[1 - \left(\frac{8}{4.16}\right)^2\right]$

z_3 $\quad E = V_0\left[1 - \left(\frac{8}{6.8}\right)^2\right]$