4: birefringence and phase matching

Polarization states in EM
Linear anisotropic response
$\chi^{(1)}$ tensor and its symmetry properties
Working with the index ellipsoid: angle tuning
Phase matching in crystals

Polarization in EM

Plane wave state, arbitrary direction: $\mathbf{E}(r,t) = \mathbf{E_0} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

In vacuum, $\mathbf{D} = \varepsilon_0 \mathbf{E}$

$$\vec{\nabla}\cdot\mathbf{E} = 0 \rightarrow \begin{pmatrix} \partial_x & \partial_y & \partial_z \end{pmatrix} \bullet \mathbf{E_0} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

$$\vec{\nabla}\cdot\mathbf{E} = 0 \rightarrow \begin{pmatrix} \partial_x & \partial_y & \partial_z \end{pmatrix} \bullet \begin{pmatrix} E_{x0} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} & E_{y0} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} & E_{z0} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \end{pmatrix}$$

$$\vec{\nabla}\cdot\mathbf{E} = \partial_x E_{x0} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \partial_y E_{y0} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \partial_z E_{z0} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

$$\partial_x E_{x0} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = E_{x0}\partial_x e^{i(k_x x + k_y y + k_z z - \omega t)} = i k_x E_{x0} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

$$\vec{\nabla}\cdot\mathbf{E} = i\left(k_x E_{x0} + k_y E_{y0} + k_z E_{z0}\right) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = i\mathbf{k}\cdot\mathbf{E_0} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = 0$$

From this we can say that $\mathbf{k}\cdot\mathbf{E_0} = 0$ and $\mathbf{k}\perp\mathbf{E}$

Therefore, the electric field lies in a plane perpendicular to k

The polarization direction can take on any linear combination of horizontal and vertical states (this includes circular polarization).

Other vector relations

Similarly,

$$\vec{\nabla}\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \rightarrow i\mathbf{k}\times\mathbf{E} = +i\omega\mathbf{B} \qquad \text{so} \quad \mathbf{B}\perp\mathbf{k},\mathbf{E}$$

Energy flow is given by the Poynting vector:

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \qquad \text{so} \quad \mathbf{S}\perp\mathbf{E},\mathbf{B}$$

and $\mathbf{S}\,||\,\mathbf{k}$

These relations hold in any isotropic medium. But if the medium is anisotropic, the vector relations must be modified.

$$\vec{\nabla}\cdot\mathbf{E} = 0 \qquad \vec{\nabla}\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

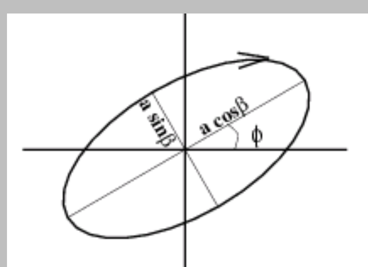
$$\vec{\nabla}\cdot\mathbf{B} = 0 \qquad \vec{\nabla}\times\mathbf{B} = \mu_0\frac{\partial \mathbf{D}}{\partial t}$$

General elliptical polarization state

For light propagating in the z-direction:

$$\vec{E}_x(z,t) = E_{0x}\cos(kz - \omega t)\,\vec{x}$$
$$\vec{E}_y(z,t) = E_{0y}\cos(kz - \omega t + \varepsilon)\,\vec{y}$$

Pick a plane (z = 0), plot E-field as f(t)



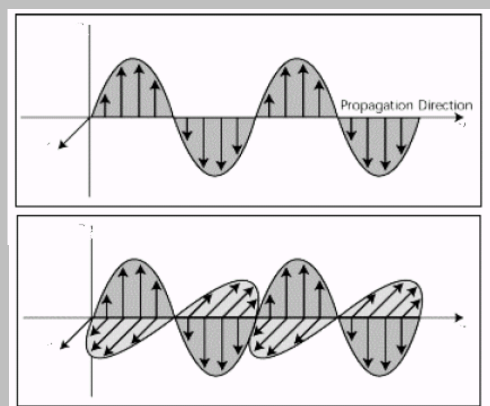
An ellipse can be represented by 4 quantities:

1. size of minor axis
2. size of major axis
3. orientation (angle)
4. sense (CW, CCW)

Linear polarization state

For light propagating in the z-direction, ε=0:

$$\vec{E}_x(z,t) = E_{0x}\cos(kz-\omega t)\,\vec{x}$$
$$\vec{E}_y(z,t) = E_{0y}\cos(kz-\omega t)\,\vec{y}$$
$$\rightarrow \begin{pmatrix} E_{0x} \\ E_{0y} \end{pmatrix}$$



B. Linearly Polarized Light at 45 Degrees

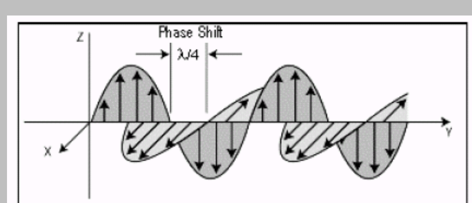
Vertical $E_0\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

45° $E_0\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

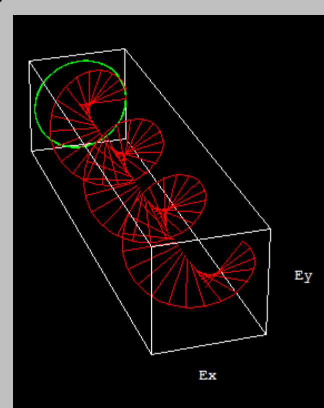
Circular polarization state

For light propagating in the z-direction, ε=±π/2:

$$\vec{E}_x(z,t) = E_{0x}\cos(kz-\omega t)\,\vec{x}$$
$$\vec{E}_y(z,t) = E_{0y}\cos(kz-\omega t \pm \pi/2)\,\vec{y}$$



C. Circularly Polarized Light



$$\vec{E}_x(z,t) = E_{0x}e^{i(kz-\omega t)}\,\vec{x}$$
$$\vec{E}_y(z,t) = E_{0y}e^{i(kz-\omega t\pm\pi/2)}\,\vec{y} = E_{0y}(\pm i)e^{i(kz-\omega t)}\,\vec{y}$$

$$E_0\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

Jones vector representation

Use when light is fully polarized, coherent

$E_0\begin{pmatrix} a \\ b \end{pmatrix}$ With *a, b* complex, normalized: $a^2+b^2=1$

Horizontal: $E_0\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ 45°: $E_0\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Vertical: $E_0\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ R/L circular: $E_0\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$

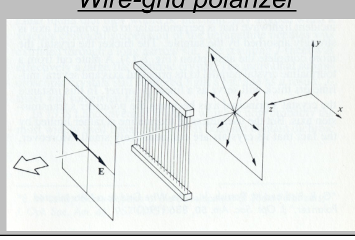
Other representation
Fields: Stokes vectors (4 components)
Optical components: Mueller matrices (4x4)
- Good for partially polarized light
- Does not track phase info for coherent light

Manipulating polarization: polarizers

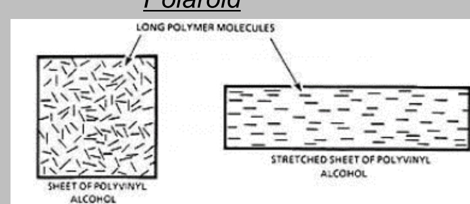
S: e-field is in plane of surface (skim)
P: e-field has out of plane component (plunge)

Effects to work with
Brewster angle: S reflects some, P does not reflect
Total internal reflection (TIR):
full reflection if AOI is greater than critical angle
Linear dichroism:
material absorbs one polarization much more than other

Wire-grid polarizer



Polaroid

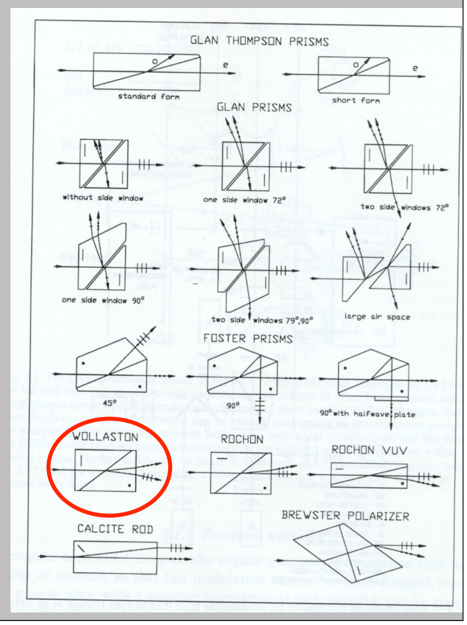


Crystal polarizers

- Crystal polarizers used as:
  - Beam displacers,
  - Beam splitters,
  - Polarizers,
  - Analyzers, ...
- Examples: Nicol prism, Glan-Thomson polarizer, Glan or Glan-Foucault prism, Wollaston prism
- Non-crystal: cube, thin-film polarizer
- Jones matrix:

Horizontal polarizer $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Vertical polarizer $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$



Retarder: half wave plate

- Retard the phase of one polarization component to change the polarization state
- half-wave plate: retard by π = 180° = λ/2
  - enter with 45° polarization relative to crystal axes

$$\mathbf{E} = E_0 \tfrac{1}{\sqrt{2}} \left(\hat{\mathbf{x}} + \hat{\mathbf{y}} \right) e^{i(kz - \omega t)} = E_0 \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

  - propagate through thickness L

$$\mathbf{E} = E_0 \tfrac{1}{\sqrt{2}} \left(\hat{\mathbf{x}} e^{ikn_o L} + \hat{\mathbf{y}} e^{ikn_e L} \right) e^{i(kz - \omega t)} = E_0 \tfrac{1}{\sqrt{2}} \begin{pmatrix} e^{ikn_o L} \\ e^{ikn_e L} \end{pmatrix} = E_0 \tfrac{1}{\sqrt{2}} e^{ikn_o L} \begin{pmatrix} 1 \\ e^{ik(n_e - n_o)L} \end{pmatrix}$$

  - set L so that $\Delta\phi = kL(n_e - n_o) = \pi + m \cdot 2\pi$ **m = order**
  - Net effect is that y component changes sign

Jones matrix for half wave plate $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Retarder: quarter wave plate

- Convert linear to circular
- quarter-wave plate: retard by π/2 = 90° = λ/4
  - enter with 45° polarization relative to crystal axes

$$\mathbf{E} = E_0 \tfrac{1}{\sqrt{2}}\left(\hat{\mathbf{x}} + \hat{\mathbf{y}}\right) e^{i(kz-\omega t)} = E_0 \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

  - now set thickness L so that

$$\Delta\phi = kL\left(n_e - n_o\right) = \tfrac{\pi}{2} + m\cdot\pi \qquad \textbf{m = order}$$

  - Net effect is that y component changes by *i*

Jones matrix for quarter wave plate $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

Rotated polarization vector

- Arbitrary orientation of linear polarization

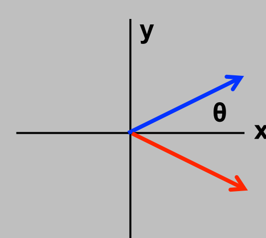
$$\mathbf{E} = E_0\left(\hat{\mathbf{x}}\cos\varphi + \hat{\mathbf{y}}\sin\varphi\right) e^{i(kz-\omega t)} = E_0 \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = E_0 \mathbf{v}_\varphi$$

- Using rotation matrix

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \rightarrow M_\varphi \cdot \mathbf{v}_x = \mathbf{v}_\varphi$$

- Half wave plate:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ -\sin\varphi \end{pmatrix}$$



- polarization rotates by 2 φ !

Rotated Jones matrix

- In practice, the waveplate is rotated in its mount
- Earlier, we had $\mathbf{M}_{\lambda/2}\cdot\left(\mathbf{M}_{\varphi}\cdot\mathbf{v}_{x}\right)=\mathbf{M}_{\lambda/2}\cdot\mathbf{v}_{\varphi}=\mathbf{v}_{-\varphi}$
- Multiply both sides by inverse rotation matrix $\left(\mathbf{M}_{-\varphi}\cdot\mathbf{M}_{\lambda/2}\cdot\mathbf{M}_{\varphi}\right)\cdot\mathbf{v}_{x}=\mathbf{M}_{-\varphi}\cdot\mathbf{v}_{-\varphi}=\mathbf{v}_{-2\varphi}$

$$\begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}\cos\varphi\\ -\sin\varphi\end{pmatrix}$$

$$=\begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}\cos^2\varphi-\sin^2\varphi\\ -2\sin\varphi\cos\varphi\end{pmatrix}=\begin{pmatrix}\cos2\varphi\\ -\sin2\varphi\end{pmatrix}$$

- This has the effect of rotating the matrix

$$\begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}=\begin{pmatrix}\cos2\varphi & \sin2\varphi\\ \sin2\varphi & -\cos2\varphi\end{pmatrix}$$

Maxwell's Equations: linear anisotropic medium

- The induced polarization, **P**, contains the effect of the medium:

$$\vec{\nabla}\cdot\mathbf{D}=0 \qquad \vec{\nabla}\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

$$\vec{\nabla}\cdot\mathbf{B}=0 \qquad \vec{\nabla}\times\mathbf{B}=\mu_0\frac{\partial\mathbf{D}}{\partial t}$$

Define the displacement vector

$$\mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P}$$

- In an anisotropic medium:

$$\mathbf{P}\left(\mathbf{E}\right)=\varepsilon_0\vec{\vec{\chi}}\cdot\mathbf{E},\quad \mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P}=\varepsilon_0\left(1+\vec{\vec{\chi}}\right)\cdot\mathbf{E}=\varepsilon_0\vec{\vec{\varepsilon}}\cdot\mathbf{E}$$

So now **D** and **E** are not necessarily parallel.

$$\mathbf{D}=\varepsilon_0\vec{\vec{\varepsilon}}\cdot\mathbf{E}\rightarrow\begin{pmatrix}D_x\\ D_y\\ D_z\end{pmatrix}=\varepsilon_0\begin{pmatrix}\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz}\\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz}\\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}\end{pmatrix}\begin{pmatrix}E_x\\ E_y\\ E_z\end{pmatrix}$$

Linear anisotropic response

For anisotropic linear response: $D_i = \varepsilon_0 \sum_j \varepsilon_{ij} E_j = \varepsilon_0 \varepsilon_{ij} E_j$

Contracted notation
Repeated indices are summed

In a basis aligned with the crystal axes:

$$\begin{bmatrix} D_X \\ D_Y \\ D_Z \end{bmatrix} = \begin{bmatrix} \varepsilon_{XX} & 0 & 0 \\ 0 & \varepsilon_{YY} & 0 \\ 0 & 0 & \varepsilon_{ZZ} \end{bmatrix} \begin{bmatrix} E_X \\ E_Y \\ E_Z \end{bmatrix} = \begin{bmatrix} \varepsilon_o & 0 & 0 \\ 0 & \varepsilon_o & 0 \\ 0 & 0 & \varepsilon_e \end{bmatrix} \begin{bmatrix} E_X \\ E_Y \\ E_Z \end{bmatrix}$$

Uniaxial case: o = “ordinary”
e = “extraordinary”

D is not parallel to E if E projects onto different axes

$$\begin{bmatrix} D_X \\ D_Y \\ D_Z \end{bmatrix} = \begin{bmatrix} \varepsilon_o & 0 & 0 \\ 0 & \varepsilon_o & 0 \\ 0 & 0 & \varepsilon_e \end{bmatrix} \begin{bmatrix} 0 \\ E_Y \\ E_Z \end{bmatrix} = \begin{bmatrix} 0 \\ \varepsilon_o E_Y \\ \varepsilon_e E_Z \end{bmatrix}$$

Linear tensor $\chi^{(1)}$

TABLE 1.5.1 Form of the linear susceptibility tensor $\chi^{()}$ as determined by the symmetry properties of the optical medium, for each of the seven crystal classes and for isotropic materials. Each nonvanishing element is denoted by its cartesian indices

Triclinic	$\begin{bmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{bmatrix}$	
Monoclinic	$\begin{bmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ zx & 0 & zz \end{bmatrix}$	
Orthorhombic	$\begin{bmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{bmatrix}$	biaxial
Tetragonal Trigonal Hexagonal	$\begin{bmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{bmatrix}$	uniaxial
Cubic Isotropic	$\begin{bmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & xx \end{bmatrix}$	isotropic

The dielectric tensor ε_{ij} is *symmetric* in a nonabsorbing medium.

$\frac{\partial U}{\partial t} = -\nabla \cdot \mathbf{S}$ Continuity equation:
Rate of change of energy density = - div of power flow

$$\left.\begin{array}{l}\nabla \cdot \mathbf{S} = \nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \ \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}\end{array}\right\} \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \dot{\mathbf{D}} - \mathbf{H} \cdot \dot{\mathbf{B}}$$

$$\Rightarrow \dot{U} = \dot{U}_E + \dot{U}_H = -\nabla \cdot \mathbf{S} = \mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{H} \cdot \dot{\mathbf{B}}$$

Look at the E component of the energy density: $\dot{U}_E = \mathbf{E} \cdot \dot{\mathbf{D}} = E_i \varepsilon_{ij} \dot{E}_j$

But we also know that: $U_E = \frac{1}{2} \mathbf{E} \cdot \mathbf{D} = \frac{1}{2} \varepsilon_{ij} E_i E_j$

Take the time derivative: $\dot{U}_E = \frac{1}{2} \varepsilon_{ij} (\dot{E}_i E_j + E_i \dot{E}_j) = \frac{1}{2} (\varepsilon_{ji} + \varepsilon_{ij}) E_i \dot{E}_j$

Therefore, the dielectric tensor is symmetric: $\boxed{\varepsilon_{ji} = \varepsilon_{ij}}$

This is an example of an intrinsic symmetry.
This does not require the symmetry of the crystal, or the linearity of the response.

The index ellipsoid

Energy density inside the medium: $U_E = \frac{1}{2} \mathbf{D} \bullet \mathbf{E} = \frac{1}{2} \varepsilon_0 \sum_{ij} \varepsilon_{ij} E_i E_j$
see Davis 18.3 for derivation

The index ellipsoid is a surface of constant energy density
(in the crystal basis):

$$U_E = \frac{1}{2\varepsilon_0} \left(\frac{D_X^2}{\varepsilon_{XX}} + \frac{D_Y^2}{\varepsilon_{YY}} + \frac{D_Z^2}{\varepsilon_{ZZ}} \right) \rightarrow \frac{1}{2\varepsilon_0 U} \left(\frac{D_X^2}{\varepsilon_{XX}} + \frac{D_Y^2}{\varepsilon_{YY}} + \frac{D_Z^2}{\varepsilon_{ZZ}} \right) = 1$$

Write this with new variables to make the ellipse equation more clear:

$$X = \left(\frac{1}{2\varepsilon_0 U_E} \right)^{1/2} D_X \quad \text{etc.} \qquad 1 = \frac{X^2}{\varepsilon_{XX}} + \frac{Y^2}{\varepsilon_{YY}} + \frac{Z^2}{\varepsilon_{ZZ}} = \frac{X^2}{n_o^2} + \frac{Y^2}{n_o^2} + \frac{Z^2}{n_e^2}$$

In an arbitrary basis, the ellipse equation looks like:

$$\left(\frac{1}{n^2} \right)_1 x^2 + \left(\frac{1}{n^2} \right)_2 y^2 + \left(\frac{1}{n^2} \right)_3 z^2 + 2\left(\frac{1}{n^2} \right)_4 yz + 2\left(\frac{1}{n^2} \right)_5 xz + 2\left(\frac{1}{n^2} \right)_6 xy = 1$$

The indices 1-6 are like the contracted notation we will use for second-order NLO

Wave propagation in birefringent crystals

Inside the medium, $\vec{\nabla}\cdot\mathbf{D}=0$
So $\vec{\nabla}\cdot\mathbf{D}=i\mathbf{k}\cdot\mathbf{D}=0$ and $\mathbf{k}\perp\mathbf{D}$
The wave is described by the D-field inside the medium.

If a wave is linearly polarized, *and* the **D**-field is oriented along one of the crystal axes, the wave sees only the refractive index corresponding to the direction of **D**.

$$\mathbf{D}(r,t)=\mathbf{D_0}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\rightarrow\hat{\mathbf{z}}D_z e^{i(k_x x-\omega t)},\quad k_x=\frac{\omega}{c}n_e$$

If **k** is parallel to one of the axes, but **D** is not, the input polarization can be resolved along o- and e- axes:

$$\mathbf{D}(r,t)=\mathbf{D_0}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\rightarrow\hat{\mathbf{z}}D_z e^{i\left(\frac{\omega}{c}n_e x-\omega t\right)}+\hat{\mathbf{y}}D_y e^{i\left(\frac{\omega}{c}n_o x-\omega t\right)}$$

In Jones vector notation,

$$\mathbf{D}(r,t)=D_0\begin{pmatrix}a\\b\end{pmatrix}\rightarrow D_0\begin{pmatrix}a\,e^{i\frac{\omega}{c}n_o x}\\ b\,e^{i\frac{\omega}{c}n_e x}\end{pmatrix}$$

The vector components get a relative phase shift

$$\Delta\phi=\frac{\omega}{c}(n_0-n_e)x$$

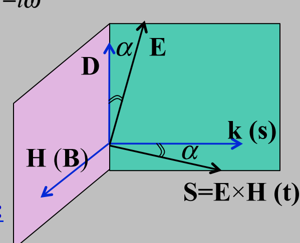
Plane wave propagation: general direction

- In an anisotropic medium, the phase velocity of light depends on its polarization state and its propagation direction.
- For a given propagation direction, there exist in general two waves, each having its own refractive index (or equivalently phase velocity) and polarization.
- All light traveling in that direction can be decomposed onto the two eigenwaves.

define wave unit vector $\mathbf{s}=\frac{\mathbf{k}}{|\mathbf{k}|}$. Then $\nabla\rightarrow i\frac{\omega n}{c}\mathbf{s},\ \frac{\partial}{\partial t}\rightarrow -i\omega$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\Rightarrow\mathbf{H}=\frac{n}{\mu c}\mathbf{s}\times\mathbf{E}$$

$$\nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t}\Rightarrow\mathbf{D}=-\frac{n}{c}\mathbf{s}\times\mathbf{H}$$



Relations between the directions of the vectors:
1) **D**, **H**, and **s** are mutually perpendicular.
2) **D**, **E**, **s**, and **E**×**H** (energy flow) lie in the same plane.
3) The Poynting vector **S**=**E**×**H** is generally not along **s**.

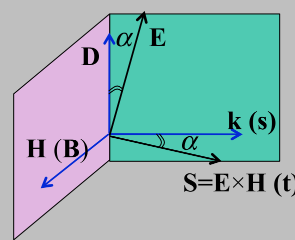
Spatial walk-off

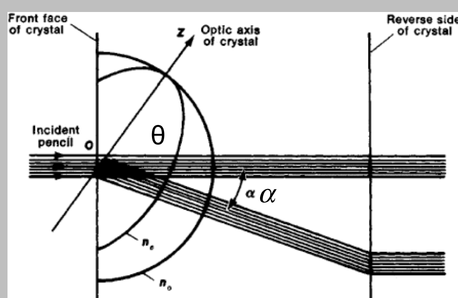
"double refraction"

Angle α between k (wavefront normal) and S (power flow)

Beam polarized along o-axis does not deviate from k

Beam at angle θ to e-axis will walk off



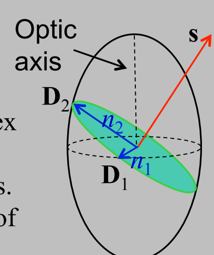


Walk-off angle:

$$\tan\alpha = \pm \frac{n_o^2}{n_e^2}\tan\theta$$

Use + for negative unixial
– for positive uniaxial

Using the index ellipsoid: tuning the refractive index



The role of the index ellipsoid:
For a given arbitrary wave normal direction **s**, the index ellipsoid can be used to
1) Find the indices of refraction of the two eigenwaves.
2) Find the corresponding directions of the **D** vectors of the two eigen waves.

The **prescription** is as follows:
1)Draw a plane that is through the origin and is perpendicular to **s**. This plane intersects the index ellipsoid surface with a particular *intersection ellipse*.
2)The lengths of the two semiaxes of the intersection ellipse, n_1 and n_2, are the two indices of refraction of the eigenwaves.
3)The two axes of the intersection ellipse are each parallel to the allowed **D** vectors of the eigenwaves.

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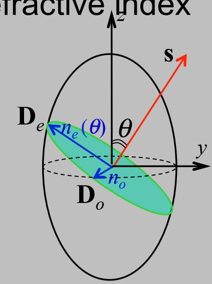
Computation of the angle-dependent refractive index

The index ellipsoid:

The equation of the index ellipsoid of a uniaxial crystal is



$$\frac{x^2+y^2}{n_o^2}+\frac{z^2}{n_e^2}=1$$

$n_o=\sqrt{\varepsilon_x/\varepsilon_0}=\sqrt{\varepsilon_y/\varepsilon_0}$, ordinary refractive index

$n_e=\sqrt{\varepsilon_z/\varepsilon_0}$, extraordinary refractive index

$n_e>n_o$: positive uniaxial crystal $\rightarrow$ prolate spheroid

$n_e<n_o$: negative uniaxial crystal $\rightarrow$ oblate spheroid

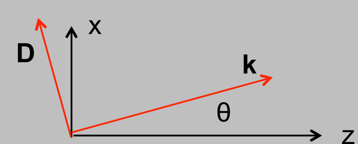
The uniaxial index ellipsoid is rotationally symmetric around the z-axis. Let **s** be in the y-z plane with a polar angle q. The two polarization directions of the **D** vectors are: $\mathbf{D}_o$ is parallel to the x-axis, $\mathbf{D}_e$ is in the y-z plane and is perpendicular to **s**.

The corresponding refractive indices are:

$$\begin{cases} n_o=n_o, \\ \frac{\left[n_e(\theta)\cos\theta\right]^2}{n_o^2}+\frac{\left[n_e(\theta)\sin\theta\right]^2}{n_e^2}=1 \Rightarrow n_e(\theta)=\left(\frac{\cos^2\theta}{n_o^2}+\frac{\sin^2\theta}{n_e^2}\right)^{-1/2}. \end{cases}$$

When **s** is on the z direction, $n_e(0°)=n_o$. Therefore the z-axis is the optic axis.

A derivation of the angle-dependent refractive index



Let $\mathbf{k}=\hat{\mathbf{x}}k\sin\theta+\hat{\mathbf{z}}k\cos\theta$

$k=k_0n_e(\theta)$

Find the components of **D** that correspond to **k**

$$\nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t}\Rightarrow -i\omega\mathbf{D}=i\mathbf{k}\times\mathbf{H}$$

$$\mathbf{k}\times\mathbf{H}=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ k_x & 0 & k_z \\ 0 & H_0 & 0\end{vmatrix}=-\hat{\mathbf{x}}k_zH_0+\hat{\mathbf{z}}k_xH_0$$

$$D_x=\frac{k_zH_0}{\omega}=\frac{H_0}{\omega}k_0n_e(\theta)\cos\theta=\frac{H_0}{c}n_e(\theta)\cos\theta$$

$$D_z=-\frac{k_xH_0}{\omega}=-\frac{H_0}{\omega}k_0n_e(\theta)\sin\theta=-\frac{H_0}{c}n_e(\theta)\sin\theta$$

A derivation of the angle-dependent refractive index

Put these into equation for ellipsoid: $\frac{1}{2\varepsilon_0 U_E}\left(\frac{D_X^2}{\varepsilon_{XX}}+\frac{D_Y^2}{\varepsilon_{YY}}+\frac{D_Z^2}{\varepsilon_{ZZ}}\right)=1=\frac{1}{2\varepsilon_0 U_E}\left(\frac{D_X^2}{n_o^2}+\frac{D_Y^2}{n_o^2}+\frac{D_Z^2}{n_e^2}\right)$

$$\frac{1}{2\varepsilon_0 U_E}\frac{H_0^2}{c^2}n_e^2(\theta)\left(\frac{\cos^2\theta}{n_o^2}+\frac{\sin^2\theta}{n_e^2}\right)=1$$

Magnetic energy density $U_H=\frac{\mu_0 H_0^2}{2}$ Is equal to electric energy density $U_H=U_E$

$$\frac{1}{2\varepsilon_0 U_E}\frac{H_0^2}{c^2}=\frac{1}{2\varepsilon_0 U_E}\frac{2U_H}{\mu_0 c^2}=\frac{U_H}{U_E}\frac{1}{\mu_0\varepsilon_0 c^2}=1$$

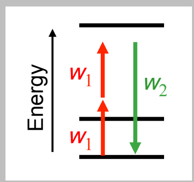
Finally: $n_e^2(\theta)=\left(\frac{\cos^2\theta}{n_o^2}+\frac{\sin^2\theta}{n_e^2}\right)^{-1}$

The refractive index can be angle-tuned anywhere between n_e and n_o.

Phase matching: an important application for the the angle-dependent refractive index

Recall that for SHG:

$$\frac{\partial A_2}{\partial z}e^{i(k_2 z-\omega_2 t)}=i\frac{\omega_2^2 d}{k_2 c^2}A_1^2 e^{i(2k_1 z-2\omega_1 t)}$$

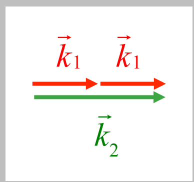


Energy *must* be conserved:

$$\omega_1+\omega_1=\omega_2 \quad\Rightarrow\quad \omega_2=2\omega_1$$

Momentum ***may or may not*** be conserved:

$$\frac{\partial A_2}{\partial z}=i\frac{\omega_2^2 d}{k_2 c^2}A_1^2 e^{i\Delta k z}$$ where $\Delta k=2k_1-k_2$



Conversion will be most efficient if $\Delta k=0$

$$\Rightarrow\quad 2\frac{\omega_1}{c_0}n(\omega_1)=\frac{2\omega_1}{c_0}n(2\omega_1)$$

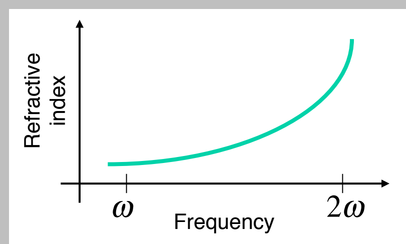
$n(\omega_1)=n(2\omega_1)$ This is the ***phase-matching condition*** for SHG

Phase-matching Second-Harmonic Generation

The phase-matching condition for SHG:

$$n(\omega) = n(2\omega)$$

Unfortunately, dispersion prevents this from ever happening!



First Demonstration of Second-Harmonic Generation

•P.A. Franken, et al, Physical Review Letters 7, p. 118 (1961)

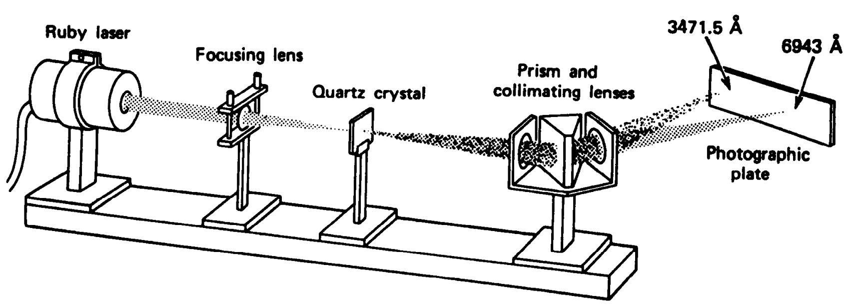


Figure 12.1. Arrangement used in the first experimental demonstration of second-harmonic generation [1]. A ruby-laser beam at $\lambda = 0.694\ \mu m$ is focused on a quartz crystal, causing the generation of a (weak) beam at $\frac{1}{2}\lambda = 0.347\ \mu m$. The two beams are then separated by a prism and detected on a photographic plate.

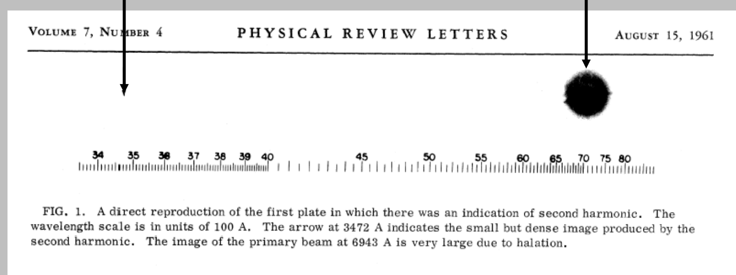
The second-harmonic beam was very weak because the process wasn't phase-matched.

First demonstration of SHG: The Data

The actual published result…

The second harmonic

Input beam



Note that the very weak spot due to the second harmonic is missing. It was removed by an overzealous Physical Review Letters editor, who thought it was a speck of dirt.

SHG without phase-matching

Non-depleted pump approximation: treat A_1 as constant

$$\frac{\partial A_2}{\partial z} = i\frac{\omega_2^2 d}{k_2 c^2} A_1^2 e^{i\Delta k z} \qquad \text{Integrate:} \quad A_2(L) = i\frac{\omega_2^2 d}{k_2 c^2} A_1^2 \int_0^L e^{i\Delta k z}\, dz$$

$$A_2(L) = i\frac{\omega_2^2 d}{k_2 c^2} A_1^2 L \frac{\left(e^{i\Delta k L} - 1\right)}{i\Delta k L}$$

Convert to intensity $I_2 = 2\varepsilon_0 n_2 c |A_2|^2$

$$\rightarrow \frac{1}{2\varepsilon_0 n_2 c} I_2(z) = \left(\frac{1}{2\varepsilon_0 n_1 c}\right)^2 I_1^2 \left(\frac{\omega_2 d}{n_2 c}\right)^2 L^2 \left(\frac{\sin(\Delta k L/2)}{\Delta k L/2}\right)^2$$

$$\rightarrow I_2(L) = \frac{\omega_2^2 d^2}{2\varepsilon_0 n_1^2 n_2 c^3} I_1^2 L^2 \operatorname{sinc}^2(\Delta k L/2)$$

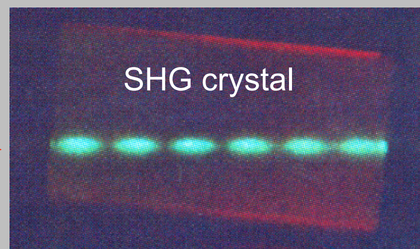
As a function of L and fixed |Δk|>0: $I_2(L) = \frac{\omega_2^2 d^2}{2\varepsilon_0 n_1^2 n_2 c^3} I_1^2 \frac{4}{\Delta k^2} \sin^2(\Delta k L/2)$

Yield oscillates:

- Period = “coherence length” $L_{coh} = 2\pi/\Delta k$
- Amplitude proportional to $\max(I_2) \propto 1/\Delta k^2$

Light created in real crystals

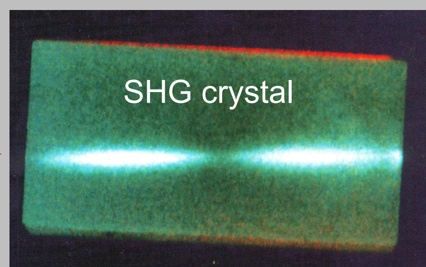
Far from phase-matching:



Input beam

Output beam

Closer to phase-matching:



Input beam

Output beam

Note that SH beam is brighter as phase-matching is achieved.

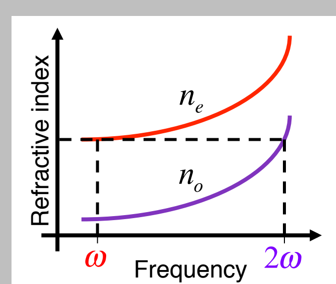
Phase-matching Second-Harmonic Generation using birefringence

Birefringent materials have different refractive indices for different polarizations. "Ordinary" and "Extraordinary" refractive indices can be different by up to 0.1 for SHG crystals.

We can now satisfy the phase-matching condition.

Put the highest frequency on the lowest index: for negative uniaxial use the extraordinary polarization for ω and the ordinary for 2ω:

$$n_e(\omega,\theta) = n_o(2\omega)$$



n_e depends on propagation angle, so we can tune for a given ω.

Some crystals have $n_e < n_o$, so the opposite polarizations work.

Real crystal dispersion data

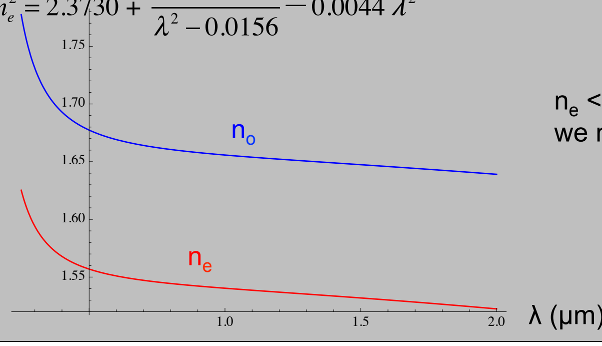
- Best resource: refractiveindex.info
- Others: crystal manufacturers, Handbook of Optics

Example: β-BBO = barium borate, BaB_2O_4

$$n_o^2 = 2.7405 + \frac{0.0184}{\lambda^2 - 0.0179} - 0.0155\,\lambda^2$$

$$n_e^2 = 2.3730 + \frac{0.0128}{\lambda^2 - 0.0156} - 0.0044\,\lambda^2$$

λ is in micrometers!



$n_e < n_o$ everywhere, so we need to angle tune

Types of phase matching

- Type 1:
  - 2ω on low index (n_e)
  - ω on high (n_o)
  - Opposite polarizations ($\chi^{(2)}$ tensor allows this)

$$\Delta k = 2\frac{\omega_1}{c} n_o(\omega_1) - \frac{\omega_2}{c} n_e(\omega_2,\theta) = 2\frac{\omega_1}{c}\left(n_o(\omega_1) - n_e(\omega_2,\theta)\right)$$

- Type 2:
  - 2ω on low index (n_e)
  - Project E_1 equally on both axes (n_o and n_e)

$$\Delta k = \frac{\omega_1}{c} n_o(\omega_1) + \frac{\omega_1}{c} n_e(\omega_1,\theta) - \frac{\omega_2}{c} n_e(\omega_2,\theta)$$

- Type 3:
  - “non-critical” or “90°” phase matching
  - Temperature-tuned
  - Only for particular crystals and wavelengths

$$\Delta k = 2\frac{\omega_1}{c}\left(n_o(\omega_1,T) - n_e(\omega_2,90,T)\right)$$

Practical issues

- Phase matching bandwidth
  - Type 1 has more BW, choose L of crystal
- Group velocity walk-off (for short pulses)
- Angular acceptance
- Birefringent beam walk-off
- Strength of nonlinearity
- Crystal damage threshold
- Thermal stability:
  - typically angle-tuned, temperature stabilized
- Available size of crystals, $$