MATH348: ABSTRACT INNER-PRODUCT SPACES

Once men turned their thinking over to machines in the hope that this would set them free. But that only permitted other men with machines to enslave them.

1. Introduction

The concept of a trigonometric series was first studied by Euler, d'Alembert and Bernoulli[1] but gained increased recognition after Joseph Fourier published his *Treatise on the propagation of heat in solid bodies* in 1807. Written in this article is,

$$\varphi(y) = a\cos\frac{\pi y}{2} + a'\cos 3\frac{\pi y}{2} + a''\cos 5\frac{\pi y}{2} + \cdots .$$

Multiplying both sides by $\cos(2k+1)\frac{\pi y}{2}$*, and then integrating from* $y = -1$ *to* $y = +1$ *yields:*

$$a_k = \int_{-1}^{1} \varphi(y)\cos(2k+1)\frac{\pi y}{2}\, dy.$$

which are now, or will be soon, familiar statements. The first point, I would like to make is that the concept of a Fourier series is quite old. In fact, the Greeks explored sinusoidal decomposition of periodic functions to explain planetary motion. However, in light of Kepler's work in the early 1600's, the exploration of the Greeks was faulted from inception. Regardless, the point I would like to make in this problem is that the rich mathematical structure,uncovered in the early 19^{th}–century, bares a striking resemblance to well-known concepts when considered in abstraction.

First, let's recall that for $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^2$ we have $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \mathbf{y} \in \mathbb{R}^2$, which is a statement that two–space is closed under linear combination. Tacit in this statement is the notion of vector addition, $+$, and scalar multiplication, $\cdot$. If we keep from dwelling too much on the mathematical particulars, then we have something that is called a *vector–space*, which is a collection of mathematical elements called *vectors*, that obey certain rules for their addition and multiplication by scalars.[2] What is important here is that any set of mathematical objects that obey the rules of a vector–space and are closed under these rules, meaning that application of these rules does not produce an element outside of the original space, can be called a vector space. For instance, if you were to consider the set of all polynomials:

1. If we multiply a polynomial by a constant then it is still a polynomial.
2. If we add polynomials together then the result is a polynomial.

Thus, without rigor, we take the set of all polynomials to form a vector–space. The same is not true of power–functions; they are not closed under the addition of power–functions.

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[1] Specifically, Daniel.

[2] An important theoretical consideration is exactly what these rules are that the vector should obey. Since they are intuitive and can be found in the definition subsection of `http://en.wikipedia.org/wiki/Vector_space`.

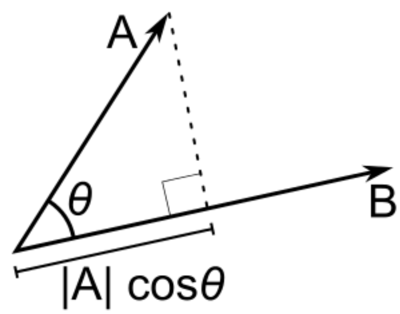
Second, we know that there is a dot–product, which takes two vectors from $\mathbb{R}^2$ and produces a scalar. Specifically, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ we have the dot–product, $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 \in \mathbb{R}$, which must obey the following rules for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$:

1. Commutativity: $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$
2. Additivity: $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$
3. Positivity: $\mathbf{x} \cdot \mathbf{x} \geq 0$ and $\mathbf{x} \cdot \mathbf{x} = 0 \iff \mathbf{x} = \mathbf{0}$

Since a dot–product of a vector with itself is never negative, we can, without imagination, take a square–root. This allows us to define the notion of length, $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + x_2^2}$. All of this can be associated to a visualizable geometry by proving that [3]

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos(\theta)$$

where



and we have that $\mathbf{A} \cdot \mathbf{B} = 0 \implies \mathbf{A} \perp \mathbf{B}$. Lastly, if we define the scalar–projection of $\mathbf{A}$ onto $\mathbf{B}$, $\mathbf{A}_B = |\mathbf{A}| \cos(\theta)$, then we can notice that $\mathbf{A}_B = (\mathbf{A} \cdot \mathbf{B})/|\mathbf{B}|$. This tells us that the dot–product is not only trying to tell us about the angle between two vectors but also how much one vector points in the direction of another. These concepts, though abstract in nature, are at the heart of Fourier series.

2. Fourier series, Inner-Products and Fourier Coefficients

Denote the set of all *reasonable* $2L$–periodic functions as $\mathcal{P}_{2L}$. This set, where $+$ means addition of functions and $\cdot$ means multiplication of a function by a scalar, is closed with respect to these operations and thus constitutes a vector space. This space is also endowed with an inner-product,

$$\langle f, g \rangle = \int_a^b f(x) g(x) dx, \quad b - a = 2L > 0. \tag{1}$$

A basis for this space is given by,

$$\mathcal{B} = \left\{ 1, \cos\left(\frac{\pi}{L}x\right), \sin\left(\frac{\pi}{L}x\right), \cos\left(\frac{2\pi}{L}x\right), \sin\left(\frac{2\pi}{L}x\right), \dots \right\}. \tag{2}$$

Thus, if $f \in \mathcal{P}_{2L}$ then there exist a_0, a_n, b_n such that

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right), \tag{3}$$

[3] The proof of this matter can be found in most linear algebra or calculus texts.

which is called the Fourier series representation of f.[4] Since the basis elements obey the orthogonality relations,

$$\langle \cos(\omega_n x), \cos(\omega_m x)\rangle = \langle \sin(\omega_n x), \sin(\omega_m x)\rangle = L\delta_{nm}, \tag{4}$$

$$\langle \cos(\omega_n x), \sin(\omega_m x)\rangle = 0, \tag{5}$$

where $\omega_n = n\pi/L$ and $n, m = 1, 2, 3, \ldots$, we find the Fourier coefficients,

$$a_0 = \frac{1}{2L}\langle f, 1\rangle \tag{6}$$

$$a_n = \frac{1}{L}\langle f, \cos(\omega_n x)\rangle, \tag{7}$$

$$b_n = \frac{1}{L}\langle f, \sin(\omega_n x)\rangle. \tag{8}$$

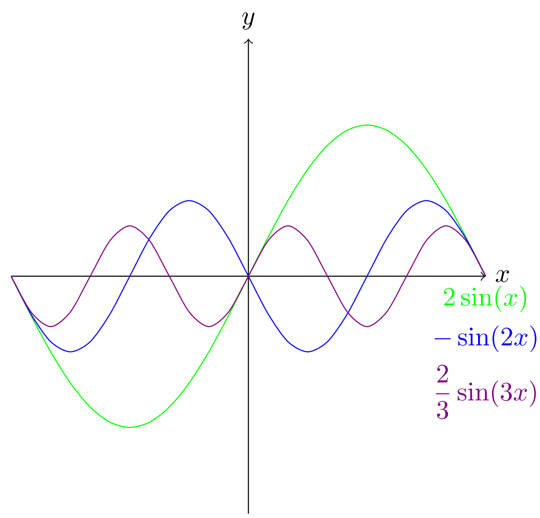
The procedure calls for us to find these coefficients when given an $f \in \mathcal{P}_{2L}$. If I could summarize these results I would say,

- A point, f, in the space of reasonable $2L$–periodic functions,$\mathcal{P}_{2L}$, can be represented as a Fourier series, Eq. (3), of the basis elements in $\mathcal{B}$. The Fourier series coefficients, Eq. (6)–(8), project the data, f, onto these mutually orthogonal basis elements and tell us how much amplitude is needed for each sinusoid of frequency ω_n so that f is the interference pattern of the linear combination of the simple sinusoids in $\mathcal{B}$.

For example if you compute the Fourier coefficients for the 2π–periodic function given by repetition of, $f(x) = x$ for $x \in (-\pi, \pi)$, you find $a_0 = a_n = 0$ and

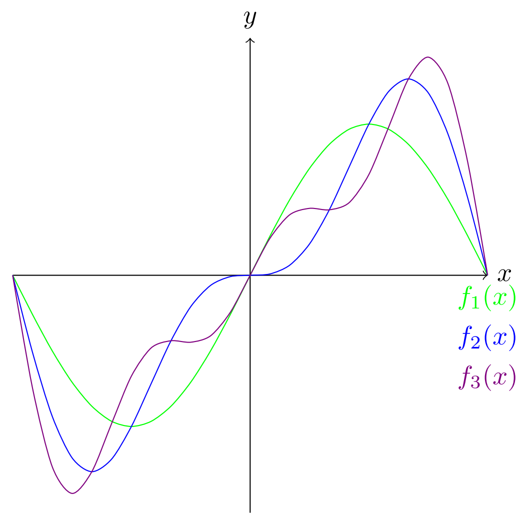
$$b_n = \frac{2}{n}(-1)^{n+1} \tag{9}$$

for $n = 1, 2, 3, \ldots$. The following graph gives us a graph of the first three modes,



and superimposing these modes together gives graphs of the partial sums,

[4] We take this on faith, which is to say that we assume that not only does this series converge but it converges to a element in $\mathcal{P}_{2L}$.



where $f_N(x) = \sum_{n=1}^{N} \frac{2}{n}(-1)^{n+1}\sin(nx)$.

3. Things to Do

1. Show that the inner–product defined by Eq. (1), satisfies the same three rules of dot–product.
2. If you have not yet derived the orthogonality relations, Eq. (4)–(5), for the Fourier series basis elements then you should.
3. If you have not yet derived the Fourier coefficients, Eq. (6)–(8), by applying the orthogonality relations, Eq. (4)–(5), to the Fourier series, Eq. (3), then you should.
4. If you could not follow the integrations from class for the sawtooth wave then you should repeat them to get the result, Eq. (9).
5. Using the sawtooth example, everyone should find the Fourier series of the function $f(x) = \alpha + x$ where $\alpha \in \mathbb{R}$ and $x \in (-\pi, \pi)$ such that $f(x + 2\pi) = f(x)$. [5]
6. Everyone should conduct a similar analysis for the function $f(x) = x^2$ where$x \in (-\pi, \pi)$ such that $f(x + 2\pi) = f(x)$.
7. Everyone should contrast this to the Fourier series of $f(x) = x^2$ where $x \in (0, 2\pi)$ such that $f(x + 2\pi) = f(x)$.
8. It would be a good idea to find the Fourier series representation of the function,

$$f(x) = \begin{cases} 0, & x \in (-2, 0), \\ x, & x \in (0, 2), \end{cases}$$

where $f(x + 4) = f(x)$.

[5] Hint: You already know half of it. Just find Fourier series representation of $f(x) = \alpha$. Hint hint: You should be able to find the Fourier series representation of $f(x) = \alpha$ in your head. Hint hint hint: If you know the Fourier series of these two functions then is the Fourier series of their sum the sum of their Fourier series. The answer is yes.

9. Lastly, find and graph <u>all</u> Fourier series representations of the function,

$$f(x) = \begin{cases} \dfrac{2k}{L}x, & x \in (0, L/2], \\ \dfrac{2k}{L}(L-x), & x \in (L/2, L), \end{cases}$$

based on the fundamental frequency $\omega_1 = \pi/L$.

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