MATH348-Advanced Engineering Mathematics		**Homework: PDEs - Part III**
Wave Equations: Traveling and Standing Waves, Nodal Lines, and Nonlinear Equations		
Text: 12.2, 12.8	Lecture Notes : N/A	Lecture Slides: N/A

Quote of Homework Six
Our vibrations were getting nasty. But why? Was there no communication in this car? Had we deteriorated to the level of dumb beasts?
Duke : Fear and Loathing in Las Vegas (1998)

1. D'alembert Solution to the Wave Equation in $\mathbb{R}^{1+1}$

Do both of these!

Show that by direct substitution the function $u(x,t)$ given by,

$$u(x,t) = \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(y)dy, \tag{1}$$

is a solution to the one-dimensional wave equation where u_0 and v_0 are the ideally elastic objects initial displacement and velocity, respectively. [1]

2. Wave Equation on a closed and bounded spatial domain of $\mathbb{R}^{1+1}$

Consider the one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad , \tag{2}$$

$$x \in (0, L)\,, \qquad t \in (0, \infty)\,, \qquad c^2 = \frac{T}{\rho}. \tag{3}$$

Equations (2)-(3) model the time-evolution of the displacement from rest, $u = u(x,t)$, of an elastic medium in one-dimension. The object, of length L, is assumed to have a homogeneous lateral tension T, and linear density ρ. That is, $T, \rho \in \mathbb{R}^+$. Assume, as well, the boundary conditions[2],

$$u_x(0,t) = 0, u_x(L,t) = 0, \tag{4}$$

and initial conditions,

$$u(x,0) = f(x), \tag{5}$$

$$u_t(x,0) = g(x). \tag{6}$$

2.1. **Separation of Variables : General Solution.** Assume that the solution to (2)-(3) is such that $u(x,t) = F(x)G(t)$ and use separation of variables to find the general solution to (2)-(3), which satisfies (4)-(6). [3] [4]

2.2. **Qualitative Dynamics.** Describe how the the general solution to (2)-(3) changes as the tension, T, is increased while all other parameters are held constant. Also, describe how the solution changes when the linear density, ρ, is increased while all other parameters are held constant.

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[1] This is called the d'Alembert solution to the wave equation. To do this you may want to recall the fundamental theorem of calculus, $\frac{d}{dx}\int_0^x f(t)dt = f(x)$ and properties of integrals, $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$.

[2] These boundary conditions imply that the object must have zero slope at each endpoint.

[3] It is important to notice that the solution to the spatial portion of the problem is the same as the heat problem above.

[4] Remember that in this case we have a nontrivial spatial solution for zero eigenvalue. From this you should find the associated temporal function should find that $G_0(t) = C_1 + C_2 t$.

2.3. **Fourier Series : Solution to the IVP.** Define,

$$f(x) = \begin{cases} \dfrac{2k}{L}x, & 0 < x \le \frac{L}{2}, \\ \\ \dfrac{2k}{L}(L-x), & \frac{L}{2} < x < L. \end{cases} \tag{7}$$

Let $L = 1$ and $k = 1$ and find the particular solution, which satisfies the initial displacement, $f(x)$, given by (7) and has zero initial velocity for all points on the object.

3. Inhomogeneous Wave Equation on a closed and bounded spatial domain of $\mathbb{R}^{1+1}$

Consider the non-homogeneous one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + F(x,t) \quad , \tag{8}$$

$$x \in (0, L)\,, \qquad t \in (0, \infty)\,, \qquad c^2 = \frac{T}{\rho}. \tag{9}$$

with boundary conditions and initial conditions,

$$u(0,t) = u(L,t) = 0, \tag{10}$$

$$u(x,0) = u_t(x,0) = 0. \tag{11}$$

Letting $F(x,t) = A\sin(\omega t)$ gives the following Fourier Series Representation of the forcing function F ,

$$F(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi x}{L}\right), \tag{12}$$

where

$$f_n(t) = \frac{2A}{n\pi}(1 - (-1)^n)\sin(\omega t). \tag{13}$$

3.1. **Educated Fourier Series Guessing.** Based on the boundary conditions we assume a Fourier sine series solution. However, the time-dependence is unclear. So, assume that,

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) G_n(t), \tag{14}$$

where $G_n(t)$ represents the unknown dynamics of the n-th Fourier mode. Using this assumption and (12)-(13), show by direct substitution that (8) yields the ODE,

$$\ddot{G}_n + \left(\frac{cn\pi}{L}\right)^2 G_n = \frac{2A}{n\pi}(1 - (-1)^n)\sin(\omega t). \tag{15}$$

3.2. **Solving for the Dynamics.** The solution to (15) is given by,

$$G_n(t) = G_n^h(t) + G_n^p(t), \tag{16}$$

where $G_n^h(t) = B_n \cos\left(\frac{cn\pi}{L}t\right) + B_n^* \sin\left(\frac{cn\pi}{L}t\right)$ is the homogeneous solution and $G_n^p(t)$ is the particular solution to (15).

3.2.1. *Particular Solution - I.* If $\omega \neq cn\pi/L$ then what would the choice for $G_n^p(t)$ be, assuming you were solving for $G_n^p(t)$ using the method of undetermined coefficients? DO NOT SOLVE FOR THESE COEFFICIENTS

3.2.2. *Particular Solution - II.* If $\omega = cn\pi/L$ then what would the choice for $G_n^p(t)$ be, assuming you were solving for $G_n^p(t)$ using the method of undetermined coefficients? DO NOT SOLVE FOR THESE COEFFICIENTS

3.2.3. *Physical Conclusions.* For the Particular Solution - II, what is $\lim_{t\to\infty} u(x,t)$ and what does this limit imply physically?

4. ~~Vibrations of a Rectangular Membrane: Wave Equation on a Bounded Domain of~~ $\mathbb{R}^{2+1}$ Ignore

Suppose that you are given an infinitesimally thin, ideally elastic membrane of area $A = L_x L_y$, which is allowed to move in the $z-$axis direction but is permanently fixed along its perimeter. Use the solution to the corresponding PDE to describe the first four fundamental vibrational modes and the structure of their nodal lines.

Advanced Engineering Mathematics		**Homework Eight**
Partial Differential Equations : Heat Equation, Wave Equation, Properties, External Forcing		
Text: 12.3-12.5	Lecture Notes : 14 and 15	Lecture Slides: 6

Quote of Homework Eight
Arrakis teaches the attitude of the knife chopping off what's incomplete and saying: "Now it's complete because it's ended here."
Frank Herbert : Dune (1965)

1. Heat Equation on a closed and bounded spatial domain of $\mathbb{R}^{1+1}$

Do this problem!

Consider the one-dimensional heat equation,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad , \tag{1}$$

$$x \in (0, L)\,, \qquad t \in (0, \infty)\,, \qquad c^2 = \frac{K}{\sigma\rho}. \tag{2}$$

Equations (1)-(2) model the time-evolution of the temperature, $u = u(x,t)$, of a heat conducting medium in one-dimension. The object, of length L, is assumed to have a homogenous thermal conductivity K, specific heat σ, and linear density ρ. That is, $K, \sigma, \rho \in \mathbb{R}^+$. If we consider an object of finite-length, positioned on say $(0, L)$, then we must also specify the boundary conditions[1],

$$u_x(0,t) = 0, u_x(L,t) = 0, . \tag{3}$$

Lastly, for the problem to admit a unique solution we must know the initial configuration of the temperature,

$$u(x, 0) = f(x). \tag{4}$$

1.1. **Separation of Variables : General Solution.** Assume that the solution to (1)-(2) is such that $u(x,t) = F(x)G(t)$ and use separation of variables to find the general solution to (1)-(2), which satisfies (3)-(4). [2]

1.2. **Qualitative Dynamics.** Describe how the long term behavior of the general solution to (1)-(4) changes as the thermal conductivity, K, is increased while all other parameters are held constant. Also, describe how the solution changes when the linear density, ρ, is increased while all other parameters are held constant.

1.3. **Fourier Series : Solution to the IVP.** Define,

$$f(x) = \begin{cases} \dfrac{2k}{L}x, & 0 < x \le \frac{L}{2}, \\[2ex] \dfrac{2k}{L}(L - x), & \frac{L}{2} < x < L \end{cases} \tag{5}$$

and for the following questions we consider the solution, u, to the heat equation given by, (1)-(2), which satisfies the initial condition given by (11). [3] For $L = 1$ and $k = 1$, find the particular solution to (1)-(2) with boundary conditions (3)-(4) for when the initial temperature profile of the medium is given by (11). Show that $\lim_{t\to\infty} u(x,t) = f_{avg} = 0.5$.[4]

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[1] Here the boundary conditions correspond to perfect insulation of both endpoints

[2] An insulated bar is discussed in examples 4 and 5 on page 557.

[3] When solving the following problems it would be a good idea to go back through your notes and the homework looking for similar calculations.

[4] It is interesting here to note that though the initial condition f doesn't appear to satisfy the boundary conditions its periodic Fourier extension does. That is, if you draw the even periodic extension of the initial condition then you would see that the slope is not well defined at the end points. Remembering that the Fourier series averages the right and left hand limits of the periodic extension of the function f at the endpoints shows that the boundary conditions are, in fact, satisfied, since the derivative of an average is the average of derivatives.

2.4. **Relation to Power-Series.** Assume that $y(x) = \sum_{n=0}^{\infty} a_n x^n$ to find the general solution of (8) in terms of the hyperbolic sine and cosine functions. [1]

3. Conservation Laws in One-Dimension

Ignore this problem

Recall that the conservation law encountered during the derivation of the heat equation was given by,

$$\frac{\partial u}{\partial t} = -\kappa \nabla \phi, \tag{10}$$

which reduces to

$$\frac{\partial u}{\partial t} = -\kappa \frac{\partial \phi}{\partial x}, \quad \kappa \in \mathbb{R} \tag{11}$$

in one-dimension of space.[2] In general, if the function $u = u(x,t)$ represents the density of a physical quantity then the function $\phi = \phi(x,t)$ represents its flux. If we assume the ϕ is proportional to the negative gradient of u then, from (11), we get the one-dimensional heat/diffusion equation.[3]

3.1. **Transport Equation.** Assume that ϕ is proportional to u to derive, from (11), the convection/transport equation, $u_t + cu_x = 0 \;\; c \in \mathbb{R}$.

3.2. **General Solution to the Transport Equation.** Show that $u(x,t) = f(x - ct)$ is a solution to this PDE.

3.3. **Diffusion-Transport Equation.** If both diffusion and convection are present in the physical system then the flux is given by, $\phi(x,t) = cu - du_x$, where $c, d \in \mathbb{R}^+$. Derive from, (11), the convection-diffusion equation $u_t + cu_x - du_{xx} = 0$.

3.4. **Convection-Diffusion-Decay.** If there is also energy/particle loss proportional to the amount present then we introduce to the convection-diffusion equation the term λu to get the convection-diffusion-decay equation,[4]

3.5. **General Importance of Heat/Diffusion Problems.** Given that,

$$u_t = Du_{xx} - cu_x - \lambda u. \tag{12}$$

Show that by assuming, $u(x,t) = w(x,t)e^{\alpha x - \beta t}$, (12) can be transformed into a heat equation on the new variable w where $\alpha = c/(2D)$ and $\beta = \lambda + c^2/(4D)$.[5]

4. Some Solutions to common PDE

Do this problem!

Show that the following functions are solutions to their corresponding PDE's.

4.1. **Right and Left Travelling Wave Solutions.** $u(x,t) = f(x - ct) + g(x + ct)$ for the 1-D wave equation.

4.2. **Decaying Fourier Mode.** $u(x,t) = e^{-4\omega^2 t}\sin(\omega x)$ where $c = 2$ for the 1-D heat equation.

4.3. **Radius Reciprocation.** $u(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ for the 3-D Laplace equation.

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[1] The hyperbolic sine and cosine have the following Taylor's series representations centred about $x = 0$,

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}. \tag{9}$$

It is worth noting that these are basically the same Taylor series as cosine/sine with the exception that the signs of the terms do not alternate. From this we can gather a final connection for wrapping all of these functions together. If you have the Taylor series for the exponential function and extract the even terms from it then you have the hyperbolic cosine function. Taking the hyperbolic cosine function and alternating the sign of its terms gives you the cosine function. Extracting the odd terms from the exponential function gives the same statements for the hyperbolic sine and sine functions. The reason these functions are connected via the imaginary number system is because when i is raised to integer powers it will produce these exact sign alternations. So, if you remember $e^x = \sum_{n=0}^{\infty} x^n/n!$ and $i = \sqrt{-1}$ then the rest (hyperbolic and non-hyperbolic trigonometric functions) follows!

[2] When discussing heat transfer this is known as Fourier's Law of Cooling. In problems of steady-state linear diffusion this would be called Fick's First Law. In discussing electricity u could be charge density and q would be its flux.

[3] AKA Fick's Second Law associated with linear non-steady-state diffusion.

[4] The u_{xx} term models diffusion of energy/particles while u_x models convection, u models energy/particle loss/decay. The final term should not be surprising! Wasn't the appropriate model for radioactive/exponential decay $Y' = -\alpha^2 Y$?

[5] This shows that the general PDE (12) can be solved using heat equation techniques.

4.4. **Driving/Forcing Affects.** $u(x,y) = x^4 + y^4$ where $f(x,y) = 12(x^2 + y^2)$ for the 2-D Poisson equation.

Note: The PDE in question are,

- Laplace's equation : $\triangle u = 0$
- Poisson's equation : $\triangle u = f(x,y,z)$
- Heat/Diffusion Equation : $u_t = c^2 \triangle u$
- Wave Equation : $u_{tt} = c^2 \triangle u$

and can be found on page 563 of Kryszig. The following will outline some common notations. It is assumed all differential operators are being expressed in Cartesian coordinates.[6]

- Notations for partial derivatives,

$$\frac{\partial u}{\partial x} = u_x = \partial_x u \tag{13}$$

- Nabla the differential operator,

$$\nabla = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \tag{14}$$

- Gradient of a scalar function,

$$\nabla u = \begin{bmatrix} \partial_x u \\ \partial_y u \\ \partial_z u \end{bmatrix} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \tag{15}$$

- Divergence of a vector,

$$\nabla \cdot \boldsymbol{v} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \partial_x v_1 + \partial_y v_2 + \partial_z v_3 \tag{16}$$

- Curl of a vector,

$$\nabla \times \mathbf{v} = \begin{bmatrix} \partial_y v_3 - \partial_z v_2 \\ \partial_z v_1 - \partial_x v_3 \\ \partial_x v_2 - \partial_y v_1 \end{bmatrix} \tag{17}$$

- Notations for the Laplacian,

$$\triangle u = \nabla \cdot \nabla u = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \cdot \begin{bmatrix} \partial_x u \\ \partial_y u \\ \partial_z u \end{bmatrix} \tag{18}$$

$$= \partial_{xx} u + \partial_{yy} u + \partial_{zz} u \tag{19}$$

$$= u_{xx} + u_{yy} + u_{zz} \tag{20}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{21}$$

[6]Of course others have worked out the common coordinate systems, which requires some elbow grease and the multivariate chain rule. Those interested in the results can find them at Nabla in Cylindrical and Spherical

The 1D Wave Eqn on $\mathbb{R}^{1+1}$: 9/19/12

When considering $u_{tt} = c^2 u_{xx}$ on $x \in (-\infty, \infty)$ the problem becomes more difficult b/c the domain leaves nothing to periodically Extend into. See 9.17.12

Key Point 1: The wave Eqn on $\mathbb{R}^{1+1}$ admits the general soln

$$u(x,t) = f(x-ct) + g(x+ct)$$

which can be readily verified.

Let $z_{\mp} = x \mp ct$ then

$$\frac{\partial u}{\partial t} = \frac{\partial f(z_-)}{\partial t} + \frac{\partial g(z_+)}{\partial t} =$$

$$= \frac{\partial z_-}{\partial t}\frac{\partial f}{\partial z_-} + \frac{\partial z_+}{\partial t}\frac{\partial g}{\partial z_+} = -c\frac{df}{dz_-} + c\frac{dg}{dz_+}$$

$$= -cf' + cg'$$

by a similar argument we get the relations:

$$\frac{\partial^2 u}{\partial t^2} = (-c)(-c)f'' + c \cdot c \cdot g''$$

$$\frac{\partial^2 u}{\partial x^2} = f'' + g''$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

For all x,t and $f\underline{n}$ f,g s.t. f'', g'' Exist.

Key Outcome: There ~~is~~ Exist sol$\underline{n}$ to $u_{tt} = c^2 u_{xx}$ which are the superposition of a right and left traveling wave, with speed c.

* Search Dan Russell superposition and feel lucky.

We can see that this holds even for a ~~simple~~ Fourier standing wave,

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n}\, x)\cos(c\sqrt{\lambda_n}\, t)$$

"Simple Plucked State from Class"

$$= \sum_{n=1}^{\infty} \frac{A_n}{2}\Big[\sin(\sqrt{\lambda_n}\, x - c\sqrt{\lambda_n}\, t) + \sin(\sqrt{\lambda_n}\, x + c\sqrt{\lambda_n}\, t)\Big]$$

$$= \underbrace{\sum_{n=1}^{\infty} \frac{A_n}{2}\sin(\sqrt{\lambda_n}(x-ct))}_{f(x-ct)} + \underbrace{\sum_{n=1}^{\infty} \frac{A_n}{2}\sin(\sqrt{\lambda_n}(x+ct))}_{g(x+ct)}$$

If we require that $u(x,0) = u_0(x)$

$$u_t(x,0) = V_0(x)$$

then

$$u(x,0) = f(x) + g(x) = u_0(x)$$

$$u_t(x,0) = -cf'(x) + c\,g'(x) = V_0(x)$$

$$\Rightarrow -c\left(u_0' - g'(x)\right) + c\,g'(x) = V_0(x)$$

$$\Rightarrow 2c\,g'(x) = V_0 + c\,u_0'$$

$$\Rightarrow g'(x) = \frac{V_0}{2c} + \frac{u_0'}{2}$$

$$\Rightarrow g(x) = \int_0^x \frac{V_0(s)}{2c}\,ds + \frac{u_0}{2}$$

$$\Rightarrow g(x+ct) = \int_0^{x+ct} \frac{V_0(s)}{2c}\,ds + \frac{u_0(x+ct)}{2}$$

$$\Rightarrow f(x-ct) = u_0(x-ct) - g(x-ct)$$

$$= u_0(x-ct) - \int_0^{x-ct} \frac{V_0(s)\,ds}{2c} - \frac{u_0(x-ct)}{2}$$

Thus

$$u(x,t) = f(x-ct) + g(x+ct) =$$

$$= \frac{u_0(x-ct)}{2} \underbrace{- \underbrace{\int_0^{x-ct} \frac{V_0(s)\,ds}{2c}}_{+\int_{x-ct}^{0}} + \int_0^{x+ct} \frac{V_0(s)}{2c}\,ds} + \frac{u_0(x+ct)}{2}$$

$$= \frac{u_0(x-ct) + u_0(x+ct)}{2} + \int_{x-ct}^{x+ct} \frac{V_0(s)\,ds}{2c} \quad *$$

Is the representation of the superposition of right + left traveling waves that also obeys the initial conditions.

Notes:

- One could also check (*) by direct substitution and noting that

$$u(x,0) = \frac{u_0(x) + u_0(x)}{2} + \cancelto{0}{\int_x^x \text{stuff}} = u_0(x)$$

there is a lot of calc hidden in this term.

$$u_t(x,0) = \frac{-c u_0(x) + c u_0(x)}{2} + \frac{cV_0(x)}{2c} + \underbrace{\frac{cV_0(x)}{2c}} = V_0(x)$$

9/12/2012

Given

(I) $U_{tt} = c^2 U_{xx}$, $x \in (0,L)$, $t \in (0,\infty)$, $c^2 = T/\rho$ | Local String Acceleration is proportional to local concavity

(II) $U_x(0,t) = 0$, $U_x(0,t) = 0$ | End points may move but must stay flat.

(III) $U(x,0) = f(x)$ | Initial Shape

$U_t(x,0) = g(x)$ | Initial Velocity

Step I: Notice that nothing has changed from Eqn (I) from class.

Thus, $U(x,t) = X(x)T(t) \Rightarrow$

$$\Rightarrow U_{tt} = X\ddot{T} = X''Tc^2 = U_{xx}c^2$$

$X \neq 0$, $T \neq 0 \Rightarrow$

$$\frac{\ddot{T}}{c^2 T} = \frac{X''}{X} = \underbrace{-\lambda \in \mathbb{R}}_{\text{Separation Constant}} \quad (*)$$

Key Argument: If the LHS is a fn of t and the RHS a fn of x and they must be Equal for all t,x then they must not be fn of t or x.

Key Outcome: (*) gives 2 sets of ODE

$$X'' + \lambda X = 0$$

$$\ddot{T} + c^2 \lambda T = 0$$

parameterized by λ.

Step II: Now things have changed b/c (II) are not the same as class.

$$u_x(0,t) = \left.\frac{\partial u}{\partial x}\right|_{x=0} = X'(0)T(t) = 0$$

Dynamics $\Rightarrow$ $X'(0) = 0$

$\Rightarrow$ $u_x(L,t) = 0 \Rightarrow X'(L) = 0$

thus our BVP is

$$X'' + \lambda X = 0, \quad \lambda \in \mathbb{R}$$

such that $X'(0) = 0$, $X'(L) = 0$

We have the 3 sets of general sol$\underline{n}$ that use a total of six f$\underline{n}$

$$\lambda > 0: \ X_1(x) = C_1 \sin(\sqrt{\lambda}\, x) + C_2 \cos(\sqrt{\lambda}\, x)$$

$$\lambda < 0: \ X_2(x) = C_3 \sinh(\sqrt{|\lambda|}\, x) + C_4 \cosh(\sqrt{|\lambda|}\, x)$$

$$\lambda = 0: \ X_3(x) = C_5 x + C_6$$

Now $X'(0) = 0 \Rightarrow C_1 = C_5 = 0$

$X'(L) = 0 \Rightarrow C_4 = 0$

thus,

$$\lambda > 0: \ X_1'(L) = -C_2 \sqrt{\lambda} \sin(\sqrt{\lambda}\, L) = 0$$

$$\Rightarrow \sqrt{\lambda_n} = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots$$

$$\Rightarrow X_n(x) = C_n \cos(\sqrt{\lambda_n}\, x), \quad C_n \in \mathbb{R}$$

Note

$\lambda = 0$; $X_3(x) = C_6$ always satisfies the B.C

or $X_0(x) = C_0$ [use zeros for convention]

We now have a set of spatial sol<u>n</u> and a set of angular freq. ~~and~~ So, we go back to the time problem, which remains unchanged from the class notes.

Thus,

$$T_n(t) = A_n \cos(c\sqrt{\lambda_n}\,t) + B_n \sin(c\sqrt{\lambda_n}\,t)$$

Well, there is actually 1 change and that is the introduction of $\lambda = 0$ as a freq. In this case

$$T'' + c^2 \underbrace{\cancel{\lambda}}_{0} T = T'' = 0$$

$$\Rightarrow \quad T_0(t) = A_0 + \cancel{A_1}\, B_0 t$$

<u>Key Outcome</u>: General Sol<u>n</u>

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) =$$

$$= X_0(x)\,T_0(t) + \sum_{n=1}^{\infty} \cancel{\cos}\, X_n(x)\,T_n(t) =$$

$$= A_0 + B_0 t + \sum_{n=1}^{\infty} \cos(\sqrt{\lambda_n}\,x)\left[A_n \cos(c\sqrt{\lambda_n}\,t) + B_n \sin(c\sqrt{\lambda_n}\,t)\right]$$

Key Point: This is just the same soln as before except the B.C. changed the ~~spati~~ type of spatial waves!

Note:

- $X_0(x)$ = constant can be thought of a wave with $\emptyset$ freq. or ∞-wavelength.

  Also, this is cosine of $\sqrt{\lambda_0} = \frac{0 \cdot \pi}{L}$.

Step III: This will be the same as class but now we note the relation

$$\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx \overset{\substack{\text{Even simplification}\\ \uparrow}}{=} 2\int_{0}^{L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx$$

$$= L\underbrace{\delta_{nm}}_{\text{Kronecker Delta fn}} = \begin{cases} L, & m = n \\ 0, & m \neq n \end{cases}$$

$$\Rightarrow \int_{0}^{L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx = \frac{L}{2}\delta_{nm}, \quad \text{for integer } m, n$$

Thus, $u(x,0) = f(x)$ implies

$$\int_0^L \underbrace{u(x,0)}_{f(x)} \cos\left(\frac{m\pi}{L}x\right)dx = \qquad \cos\left(\frac{m\pi}{L}x\right)dx$$

$$= \int_0^L \left[A_0 + B_0 \overset{0}{\cancel{t}} + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L}x\right)\left[A_n \underset{1}{\cancel{\cos(0)}} + B_n \overset{0}{\cancel{\sin(0)}}\right]\right] \cdot$$

$$= A_0\int_0^L \cos\left(\frac{m\pi}{L}x\right)dx + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx}_{\frac{L}{2}\delta_{nm}} =$$

$$= \underbrace{\cancel{\frac{A_0 L}{m\pi}\sin\left(\frac{m\pi}{L}x\right)\Big|_0^L}}_{0-0=0} + \sum_{n=1}^{\infty} A_n \frac{L}{2}\delta_{nm} = A_m \frac{L}{2}\overset{1}{\cancel{\delta_{mm}}}$$

$$\Rightarrow \boxed{A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx,}$$

the m was an arbitrary integer so we just call it n for ease of use.

Also,

$$\int_0^L u(x,0)\,dx = A_0 \int_0^L dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos\left(\frac{n\pi}{L}x\right)dx$$

$$= A_0 \cdot L + \sum_{n=1}^{\infty} \frac{A_n L}{n\pi} \underbrace{\sin\left(\frac{n\pi}{L}x\right)\Big|_0^L}_{0-0=0} \Rightarrow \boxed{A_0 = \frac{1}{L}\int_0^L f(x)\,dx}$$

Now, the initial velocity says,

$$\int_0^L u_t(x,0)\cos\left(\frac{m\pi}{L}x\right)dx = \qquad \cos\left(\frac{m\pi}{L}x\right)dx$$

$$= \int_0^L \left[B_0 + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L}x\right)\left[c\sqrt{\lambda_n}A_n \underbrace{\sin(0)}_{0} + B_n c\sqrt{\lambda_n}\underbrace{\cos(0)}_{1}\right]\right]$$

$$= B_0 \underbrace{\int_0^L \cos\left(\frac{m\pi}{L}x\right)dx}_{0-0=0} + \sum_{n=1}^{\infty} B_n c\sqrt{\lambda_n} \underbrace{\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx}_{\frac{L}{2}\delta_{nm}}$$

$$= B_m c\sqrt{\lambda_m}\,\frac{L}{2}\,\delta_{mm} \Rightarrow \boxed{B_n = \frac{2}{c\sqrt{\lambda_n}}\int_0^L g(x)\cos\left(\frac{n\pi}{L}x\right)dx}$$

Lastly, a similar argument gives,

$$\boxed{B_0 = \frac{1}{L}\int_0^L u_t(x,0)\,dx.}$$

Eqn (11) with all Green boxes Represents the soln to the initial boundary value Problem (I)–(III).

Notes:

- Suppose $f(x)=0$, for all x and $g(x)>0$ for all x. Then

$$A_0 = A_n = 0 \text{ for all } n,$$

and

$$B_0 > 0$$

thus the displacement grows in time. That is, the ~~str~~ spring moves "up" forever. AKA your ~~are~~ throw is it.

Notes Cont:

- T, ρ control time-freq. as before.

- There is a new Equilibrium (Rest) state where the string is initially ~~flat~~. only flat

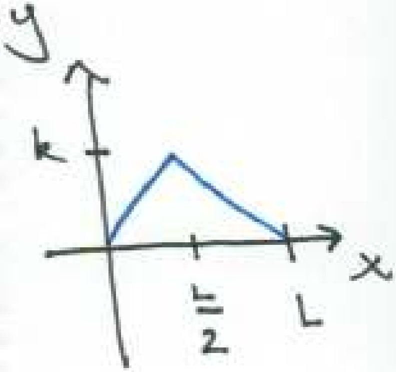
$U(x,0)=\alpha \in \mathbb{R} \Rightarrow A_n=0$ for all n.

$U_t(x,0)=0 \Rightarrow B_0=B_n=0$

$\Rightarrow U(x,t)=\alpha$ [This was always zero for our fixed end cond.]

If $g(x)=0$ for all x and

$$f(x)=\begin{cases} \frac{2k}{L}x, & x\in(0,\frac{L}{2}) \\ \frac{2k}{L}(L-x), & x\in(\frac{L}{2},L) \end{cases}$$



then $B_0=B_n=0$ and

$$A_0 = \frac{1}{L}\int_0^L u(x,0)\,dx = \frac{1}{L}\left[\frac{1}{2}\cdot L\cdot k\right] =$$

$$= \frac{k}{2}$$

$$A_n = \frac{2}{L}\int_0^L u(x,0)\cos\left(\frac{n\pi}{L}x\right)dx =$$

$$= \frac{2}{L}\left[\int_0^{L/2} \frac{2k}{L}\overset{u_1}{x}\underbrace{\cos\left(\frac{n\pi}{L}x\right)dx}_{dv} + \int_{L/2}^{L} \frac{2k}{L}\overbrace{(L-x)}^{u_2}\underbrace{\cos\left(\frac{n\pi}{L}x\right)dx}_{dv}\right]$$

$$= \frac{4k}{L^2}\left[\cancel{\frac{L}{2n\pi}\sin\left(\frac{n\pi}{2}\right)} + \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{L^2}{n^2\pi^2} - \cancel{\frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right)} + \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{L^2}{n^2\pi^2}\right]$$

u_1	u_2	dv
x +	$L-x$ +	$\cos\left(\frac{n\pi}{L}x\right)$
1 −	-1 −	$\frac{L}{n\pi}\sin\left(\frac{n\pi}{L}x\right)$
	0	$\frac{-L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{L}x\right)$

$$= \frac{8k}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right] = \begin{cases} -\dfrac{8k}{n^2\pi^2}, & n = 1,3,5,\cdots \\ 0, & n = 4,8,12,\cdots \\ -\dfrac{16k}{n^2\pi^2}, & n = 2,6,10,\cdots \end{cases}$$

Inhomogeneous PDE:

Recall: Given ~~If~~ $my'' + by' + ky = f(t)$ the general soln is of the form

$$y(t) = y_h(t) + y_p(t)$$

where $y_h(t)$ is the soln to

$$my'' + by' + ky = 0 \quad \Big\} \text{ Homogeneous Eqn.}$$

We hope something similar occurs with

$$\text{(I)} \qquad u_{tt} = c^2 u_{xx} + F(x,t), \quad x \in (0,L)$$

$$\text{(II)} \qquad u(0,t) = 0, \; u(L,t) = 0 \qquad t \in (0,\infty)$$

$$\text{(III)} \qquad u(x,0) = u_t(x,0) = 0$$

First, what is the $\overbrace{\text{homogeneous}}^{\text{for when } F(x,t)=0}$ soln?

This will guide our soln process.

Well, in that case we have a unforced string with fixed ends. The general sol$\underline{n}$ to this is given by

$$G_n^h(t) \equiv \text{Homogeneous dynamics}$$

$$u_h(x,t) = \underbrace{\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)}_{\text{Shape made by interference of spatial waves.}} \overbrace{\left[A_n \cos(c\sqrt{\lambda_n}t) + B_n \sin(c\sqrt{\lambda_n}t) \right]}^{G_n^h(t)}$$

We expect the shape $f\underline{n}$ will still be viable b/c of Fourier interference principles but the dynamics could be different. Thus, we guess

$$(*) \quad u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) G_n(t)$$

Ack, I'm using G_n out of Habit. You may want to use T_n

Where $G_n(t)$ is an unknown dynamic.

Goal: Find $G_n(t)$!

How: Sub $(*)$ into nonhomogeneous PDE.

From (I) we have,

$$u_{tt} - c^2 u_{xx} - F(x,t) =$$

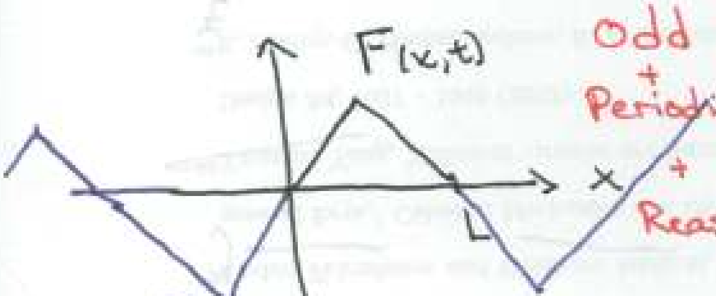
$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) G_n''(t) - c^2 \sum_{n=1}^{\infty} -\left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi}{L}x\right) G_n(t) - \underbrace{F(x,t)}_{*}$$

by linear ind. of sine f_n.

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) \overbrace{\left[G_n'' + \left(\frac{cn\pi}{L}\right)^2 G_n - \underbrace{f_n(t)}_{*}\right]}^{=0} = 0$$

<u>Key Idea:</u> We don't know what $F(x,t)$ is but it <u>must</u> be defined on only $(0,L)$. Thus, we can find an odd periodic Extension of F and therefore a Fourier Series Rep.

<u>Graphically:</u>



Odd + Periodic + Reasonable $\Rightarrow$

odd periodic wave $F^*(x,t)$.
s.t. $F^*(x,t) = F(x,t)$ for $x \in (0,L)$

<u>Math:</u>

$$F(x,t) = \sum_{n=1}^{\infty} f_n \overbrace{\sin\left(\frac{n\pi}{L}x\right)}^{\text{odd + periodic}}$$

$$f_n = \underbrace{\frac{2}{L}\int_0^L F(x,t)\sin\left(\frac{n\pi}{L}x\right)dx}_{\text{Reasonable}}$$

Thus for $u_{tt} - c^2 u_{xx} - F(x,t) = 0$ we Require

(*) $$G_n'' + \underbrace{\left(\frac{cn\pi}{L}\right)^2}_{c^2\lambda_n,\ \sqrt{\lambda_n} = \frac{n\pi}{L}} G_n = f_n, \quad n = 1, 2, 3, \dots$$

Note:

- (*) $T_n'' + c^2 \lambda_n T_n = f_n$ is the same time Eqn as before but now with an External forcing term, which results in new dynamics.

We are told to assume $f_n(t) = \frac{2A}{n\pi}\left(1 - (-1)^n\right)\sin(\omega t)$ thus,

$$G_n'' + c^2 \lambda_n G_n = \frac{2A}{n\pi}\left(1 - (-1)^n\right)\sin(\omega t)$$

whose homogeneous Soln is known as [Right?! From homogeneous fixed string.

$$G_n^h(t) = A_n \cos(c\sqrt{\lambda_n}\, t) + B_n \sin(c\sqrt{\lambda_n}\, t)$$

So, what is the particular soln $G_n^p(t)$?

Well, that depends. Since $f_n(t) \propto \sin(\omega t)$ we should guess $G_n^p(t) = \alpha_n \sin(\omega t) + \beta_n \cos(\omega t)$.

That is, unless $\omega = \frac{cn\pi}{L}$, which means that $G_n^p(t)$ is the same as $G_n^h(t)$, up to constants. In that case

$$G_n^p(t) = \alpha_n t \sin\left(\frac{cn\pi}{L}t\right) + \beta_n t \cos\left(\frac{cn\pi}{L}t\right)$$

which has the unfourtunate behavior of

$$\lim_{t\to\infty} G_n^p(t) = \infty.$$

This is resonance of a forced string! Thus, if

$$\omega \neq \frac{cn\pi}{L} \Rightarrow u(x,t) = \sum_{n=1}^{\infty} \sin\Big(\overset{\sqrt{\lambda_n}}{\frac{n\pi}{L}}x\Big) G_n(t) =$$

If $\omega = \frac{cn\pi}{L}$ in purple.

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) \underbrace{\begin{bmatrix} A_n \cos(c\sqrt{\lambda_n}t) + \\ + B_n \sin(c\sqrt{\lambda_n}t) \end{bmatrix}}_{G_n^h(t)} + \Bigg\} \text{homogeneous Soln}$$

$$\text{Particular Soln} \Bigg\{ + \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) \underbrace{\left[\alpha_n \overset{t}{} \cos(c\sqrt{\lambda_n}t) + \beta_n \overset{t}{} \sin(c\sqrt{\lambda_n}t)\right]}_{G_n^p(t)}$$

If $\omega = cn\pi/L$

If $\omega = cn\pi/L$

Via Undetermined coeff

Via Undetermined Coeff.

9/19/12

The heat Eqn:

The Equation

$$U_t = c^2 U_{xx}, \quad (\mathrm{I})$$

is called the 1D Homogeneous heat or diffusion Eqn. It is called heat b/c it was one of the first PDE studied by Fourier. However, it is a general Eqn that Evolves a density u which is controlled by the second-law of thermodynamics.

Step I: $U(x,t) = X(x)T(t) \Rightarrow$ for $X(x) \neq 0$, $T(t) \neq 0$

$$(\mathrm{I}) \Leftrightarrow \frac{T'}{c^2 T} = \frac{X''}{X} = -\lambda \in \mathbb{R}$$

$$\Rightarrow T' = -\lambda c^2 T$$

$$X'' + \lambda X = 0$$

Step 2: Recall that the spatial ODE has the sol$\underline{n}$ set

$$\lambda>0: \quad X_1(x) = C_1 \sin(\sqrt{\lambda}x) + C_2 \cos(\sqrt{\lambda}x)$$

$$\lambda<0: \quad X_2(x) = C_3 \sinh(\sqrt{|\lambda|}x) + C_4 \overset{\cosh}{\cancel{\sinh}}(\sqrt{|\lambda|}\,x)$$

$$\lambda=0: \quad X_3(x) = C_5 x + C_6$$

There are two interesting boundary conditions to consider for $x \in (0,L)$

(II) $X(0)=0, \; X(L)=0$ [See lecture 9.7.12

(II') $X'(0)=0, \; X'(L)=0$ [See lecture 9.19.12

$\underline{(\text{II})} \Rightarrow X_n(x) = \sin(\sqrt{\lambda_n}x), \; \sqrt{\lambda_n} = \frac{n\pi}{L}, \; n=1,2,3,\ldots$

$\underline{(\text{II}')} \Rightarrow X_n(x) = \cos(\sqrt{\lambda_n}x), \; \sqrt{\lambda_n} = \frac{n\pi}{L}, \; n=\underline{\underline{\underline{0}}},1,2,3$

(*) Recall if $n=0 \Rightarrow X_0(x) = 1$, which is the constant sol$\underline{n}$ for $\lambda_0=0$

Now, we ~~ser~~ return to time we have

$$T_n' = -\lambda_n c^2 T_n \quad , \; n = \underline{\underline{0}}, 1, 2, \ldots$$

If $X'(0) = 0$
$X'(L) = 0$

which asks what fn T_n produces a $-\lambda_n c^2$ multiple of itself upon 1 diff. step.

$$T_n(t) = A_n e^{-\lambda_n c^2 t} \quad , \; A_n \in \mathbb{R}, \; n = \underline{\underline{0}}, 1, 2, \ldots$$

Which gives the general soln

$$\underline{u(x,t)} = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n}\, x)\, e^{-c^2 \lambda_n t}$$

or

$$\underline{u(x,t)} = \sum_{n=0}^{\infty} A_n \cos(\sqrt{\lambda_n}\, x)\, e^{-c^2 \lambda_n t} = \underline{\underline{A_0}} + \sum_{n=1}^{\infty} A_n \cos(\sqrt{\lambda_n}\, x)\, e^{-c^2 \lambda_n t}$$

dep. on (II') or (II).

Key Points:

- $u = u(x,t)$ is a density $[u] = \frac{\text{Stuff}}{\text{length}}$
- Stuff could be:
  - Heat Energy → u = temp
  - Mass ~~Density~~ → u = density of ~~marble~~ maybe impurity
  - Probability → Prob density
- c^2 is called diffusivity, $[c^2] = \frac{\text{length}^2}{\text{time}}$ and measures how ~~the~~ much/easy stuff ~~want~~ is allowed to flow/s through the object.
- $[\lambda_n c^2] = \frac{1}{\text{length}^2} \cdot \frac{\text{length}^2}{\text{time}} = \frac{1}{\text{time}}$, decay rate.

• If we think about U as temp then:

the object touches a universe of

i) (II) $\Rightarrow$ zero temp on Relative scale

ii) (II') $\rightarrow$ the object's temp has zero slope in temp at Edges. $\Rightarrow$

$\Rightarrow$ no local ~~te~~ temp diff $\Rightarrow$ no heat flow b/c of local Equilibrium. } Ideal Insulati

• In these cases:

i) $\lim_{t\to\infty} U(x,t)=0$, the object attains Equilibrium with universe as $t\to\infty$.

ii) $\lim_{t\to\infty} U(x,t) = A_0 = \frac{1}{L}\int_0^L U(x,0)\,dx =$

$= U_{Average}(x,0)$, the object Establishes a Constan Equilibrium state that is the average of the initial temp as $t\to\infty$

- If $c^2 = \dfrac{K}{\sigma\rho}$, $K \equiv$ thermal conductivity
  $\sigma \equiv$ Specific heat
  $\rho \equiv$ density

then as K increases for fixed σ, ρ we have faster decay to Equilibrium.

Also, as ρ increases for fixed K, σ we have slower decay to Equilibrium.

- Lastly if $u(x,0) = f(x)$ then

(II) $u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n}\, x)$

$\Rightarrow A_n = \frac{2}{L}\int_0^L f(x)\sin(\sqrt{\lambda_n}\, x)\,dx$

(II') $u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\sqrt{\lambda_n}\, x)$

$\Rightarrow A_0 = \frac{1}{L}\int_0^L f(x)\,dx$, $A_n = \frac{2}{L}\int_0^L f(x)\cos(\sqrt{\lambda_n}\, x)\,dx$

So, if $f(x) = \begin{cases} \frac{2k}{L}x, & x \in \left(0, \frac{L}{2}\right) \\ \frac{2k}{L}(L-x), & x \in \left(\frac{L}{2}, L\right) \end{cases}$

(II) $A_n = \frac{8k}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$, [See 9.14.12]

(II') $A_0 = \frac{k}{2}$, $A_n = \frac{8k}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right]$ [See Problem 1 from this HW!]

Key Point:

The modes that make the triangle are the same, as they should be, but the time dynamics are not, which is expected b/c diffusion is different than ideal vibrations.

9/19/12

Soln to common PDE:

1.1: Show $u(x,t) = f(x-ct) + g(x+ct)$ is a soln to $u_{tt} = c^2 u_{xx}$. See Annotations for problem 3.

1.2:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left[e^{-4w^2 t} \underbrace{\sin(wx)}_{\text{Fourier mode}} \right] = \sin(wx) \frac{\partial}{\partial t}\left[e^{-4w^2 t} \right]$$

$$= -4w^2 e^{-4w^2 t} \sin(wx)$$

Similarly,

$$\frac{\partial u}{\partial x} = w\cos(wx) e^{-4w^2 t}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -w^2 \sin(wx) e^{-4w^2 t}$$

$$\Rightarrow \underbrace{u_t = -4w^2 e^{-4w^2 t} \sin(wx)}_{u_t} = c^2 \overbrace{\left(-w^2 \sin(wx) e^{-4w^2 t}\right)}^{u_{xx}}$$

$\Rightarrow c^2 = 4 \Rightarrow c = \pm 2 \Rightarrow u$ solves $u_t = c^2 u_{xx}$ if $c = 2$

9/19/2012

1.3: PDE is $\Delta u = u_{xx} + u_{yy} + u_{zz} = 0$

Option 1: Straight up

$$u(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}} \Rightarrow u_x = \frac{x}{(x^2+y^2+z^2)^{3/2}}$$

$$\Rightarrow u_{xx} = \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\Rightarrow u_{xx} + u_{yy} + u_{zz} = \frac{3}{(x^2+y^2+z^2)^{3/2}} - \underbrace{\frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}}}_{3(x^2+y^2+z^2)^{-3/2}} = 0$$

Option 2: Multivar. Chain Rule

$$u(x,y,z) = u(r) = \frac{1}{r}, \quad r^2 = x^2+y^2+z^2$$

$$\Rightarrow u_x(r) = u_r r_x \Rightarrow u_{xx} = u_{rx} r_x + u_r r_{xx} =$$

$$= u_{rr}(r_x)^2 + u_r r_{xx}$$

$$\Rightarrow \Delta u = u_{rr}(r_x^2 + r_y^2 + r_z^2) + u_r(r_{xx} + r_{yy} + r_{zz})$$

Where

$$r_x = \frac{1}{2} \cdot \frac{2x}{(x^2+y^2+z^2)^{1/2}} = \frac{x}{r}.$$

$$\Rightarrow r_{xx} = \frac{1}{r} - \frac{x}{r^2} \cdot r_x = \frac{1}{r} - \frac{x^2}{r^3}$$

$$\Rightarrow r_{xx} + r_{yy} + r_{zz} = \frac{3}{r} - \left(\frac{x^2+y^2+z^2}{r^3} \right) = \frac{3}{r} - \frac{r^2}{r^3} = \frac{2}{r}$$

and

$$r_x^2 + r_y^2 + r_z^2 = \frac{x^2+y^2+z^2}{r^2} = 1$$

thus in the radial variable

$$0 = \Delta u = u_{rr} + \frac{2}{r} u_r$$

and

$$u(r) = \frac{1}{r} \Rightarrow u_r = \frac{-1}{r^2}, \quad u_{rr} = +\frac{2}{r^3}$$

$$\Rightarrow u_{rr} + \frac{2}{r} u_r = \frac{2}{r^3} + \frac{2}{r} \cdot \left(\frac{-1}{r^2} \right) = 0$$

Ah, the multivar. chain rule, how I've missed you.

1.4: PDE: $\Delta u = f(x,y)$ in 2D

$$u(x,y) = x^4 + y^4 \Rightarrow u_{xx} = 4\cdot 3x^2, \quad u_{yy} = 4\cdot 3y^2$$

$$\Rightarrow u_{xx} + u_{yy} = 12(x^2+y^2) = f(x,y)$$