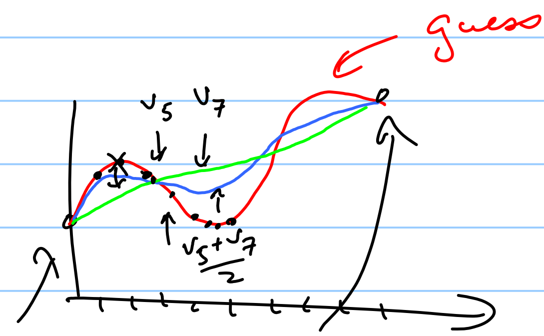
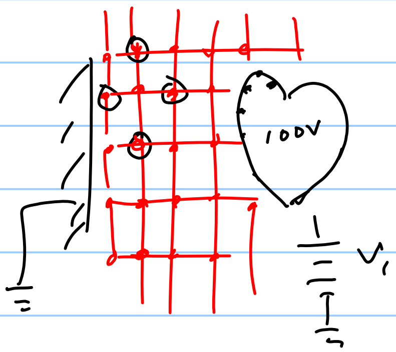
Note Title 2/1/2008

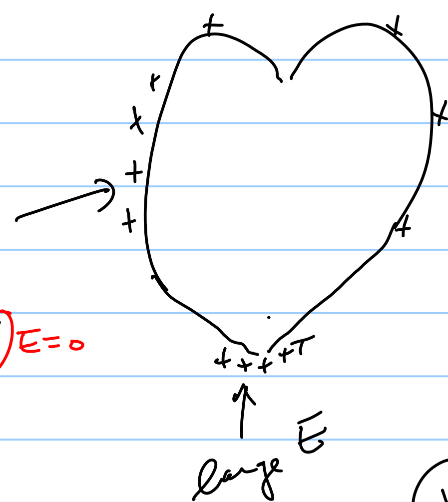
Soln to $\nabla^2 V = 0$ with bndry conditions

- method images
- sep. variables
- relaxation method

1-D

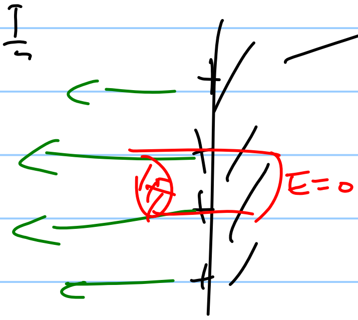






$E \cancel{4\pi\sigma} = \frac{\sigma\,\cancel{4\pi}}{\epsilon_0}$

$E_\perp = \frac{\sigma}{\epsilon_0}$

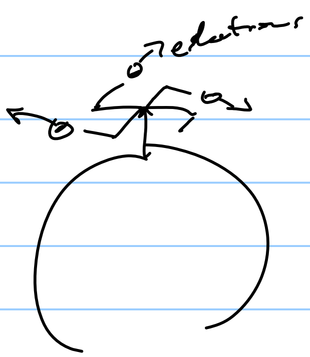


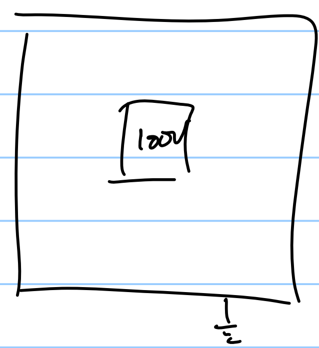
ρ

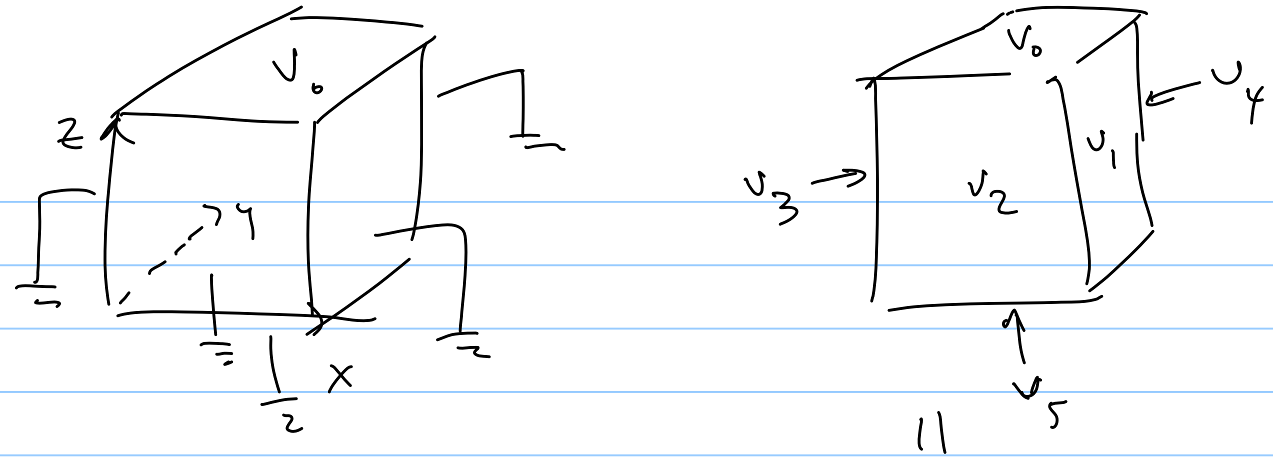
$V \rightarrow E$

$\vec{E} = -\vec{\nabla} V$

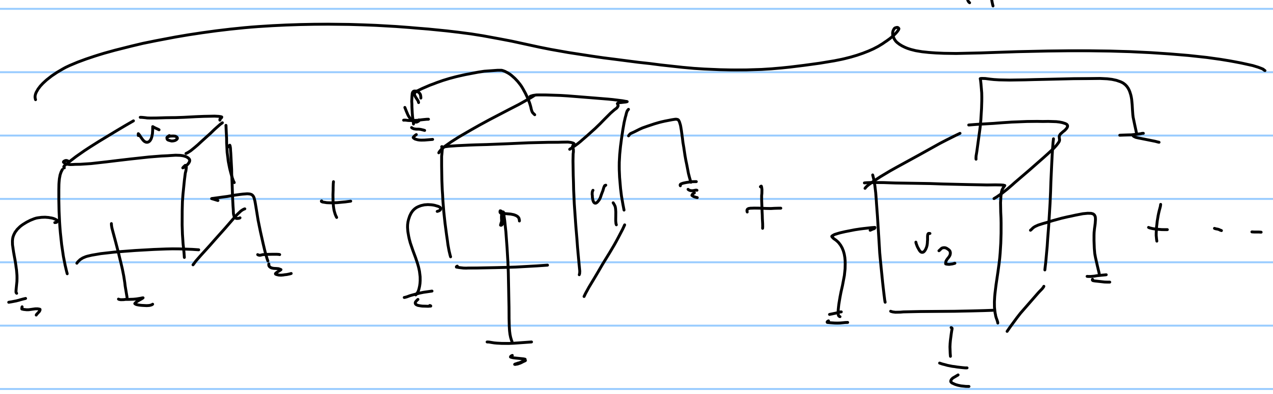
$E_\perp = -(\nabla V)_\perp = -\frac{\partial V}{\partial n}$



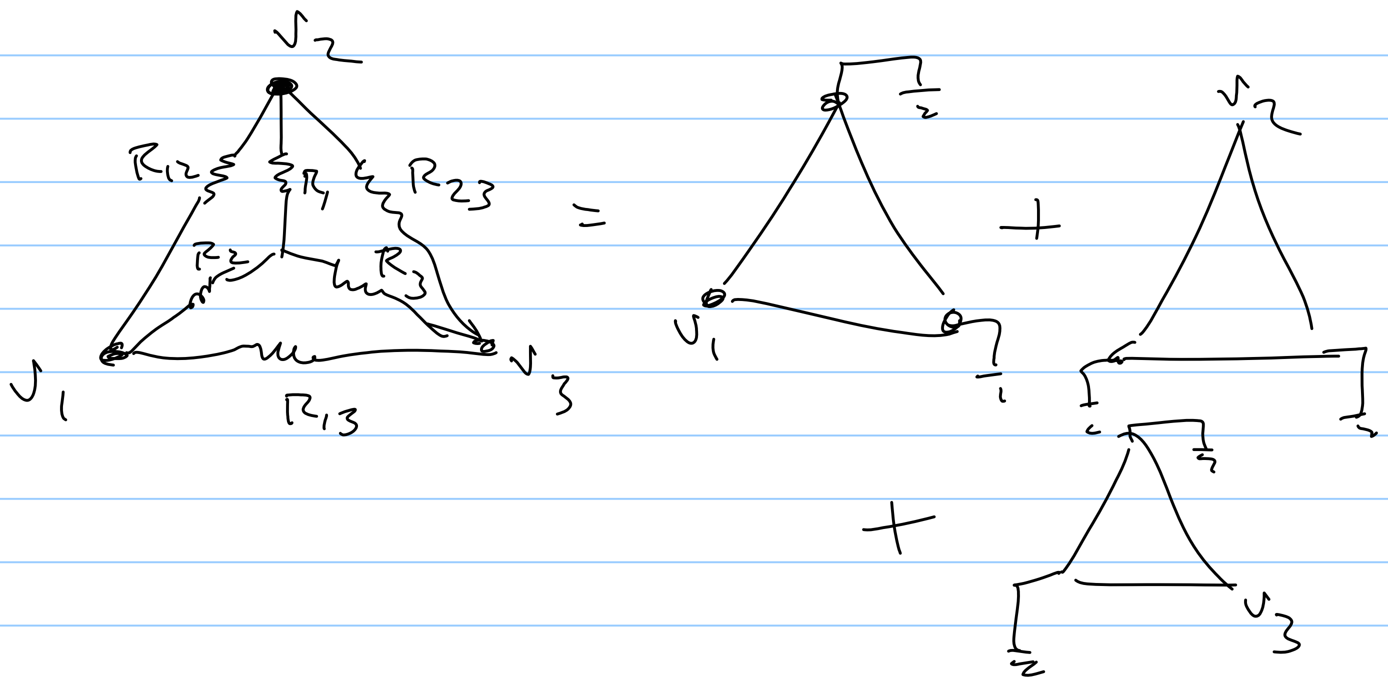




V_0
z
y
x
V_0
V_4
V_1
V_3
V_2
V_5



V_0
+
V_1
+
V_2
+ . .



V_2
R_{12}
R_1
R_{23}
R_2
R_3
V_1
V_3
R_{13}
=
V_1
+
V_2
+
V_3

Laplaces eqn: spherical coords

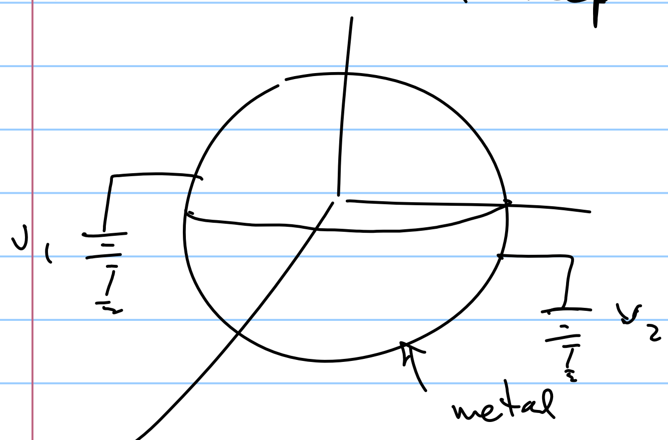
$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2} = 0$$

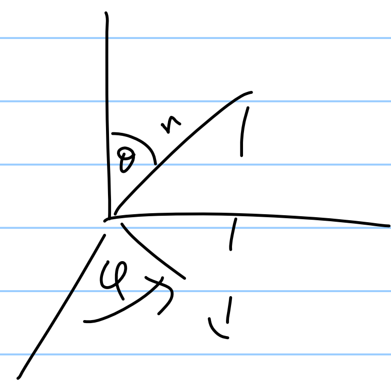
Sep. variables: assume $V(r,\theta,\varphi) = R(r)\,\Theta(\theta)\,\Phi(\varphi)$

plug into $\nabla^2 V = 0$ & divide by $V(r,\theta,\varphi)$

$$\underbrace{\frac{\sin^2\theta}{R(r)}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right)}_{C_1} + \underbrace{\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right)}_{C_2} + \underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}}_{C_3} = 0$$

assume no φ depend $\Rightarrow \Phi(\varphi) = \text{const} \Rightarrow C_3 = 0$





Laplaces eqn

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right)}_{k} + \underbrace{\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right)}_{-k} = 0$$

PDE $\Rightarrow$ 2 ODE's

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = k \qquad \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -k$$

$$\Downarrow$$

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - kR = 0$$

Two solns : r^ℓ & $r^{-(\ell+1)}$ where $k = \ell(\ell+1)$

General soln $R(r) = Ar^\ell + Br^{-(\ell+1)}$

$\Theta = P_\ell(\overbrace{\cos\theta}^{x})$ $\quad P_0 = 1$

poly in $(\underbrace{\cos\theta}_{x})$ $\quad P_1 = x = \cos\theta$

$P_2 = \frac{1}{2}(3x^2-1) = \frac{1}{2}(3\cos^2\theta - 1)$

$P_3 = \frac{1}{2}(5x^3 - 3x) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$

$$\int_{-1}^{1} P_\ell(x)P_m(x)\,dx = \int P_\ell(\cos\theta)P_m(\cos\theta)(-\sin\theta\,d\theta)$$

$= 0$ if $\ell \neq m$