3.9 Elementary operations and Gaussian Elimination

I am assuming that you've seen this before, so this is a very terse review. If not, see the book by Strang in the bibliography.

Elementary matrix operations consist of:

- [illegible]s)
- [illegible]
- [illegible]

If you have a matrix that can be derived from another matrix by a sequence of elementary operations, then the two matrices are said to be row or column equivalent. For example

$$A = \begin{pmatrix} 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 2 \\ 1 & -1 & 2 & 3 \end{pmatrix}$$

is row equivalent to

equivalent

$$B = \begin{pmatrix} 2 & 4 & 8 & 6 \\ 1 & -1 & 2 & 3 \\ 4 & -1 & 7 & 8 \end{pmatrix}$$

because we can add 2 times row 3 of A to row 2 of A; then interchange rows 2 and 3; finally multiply row 1 by 2.

Gaussian elimination consists of two phases. The first is the application of elementary operations to try to put the matrix in row-reduced form; i.e., making zero all the elements below the main diagonal (and normalizing the diagonal elements to 1). The second phase is back-substitution. Unless the matrix is very simple, calculating any of the four fundamental subspaces is probably easiest if you put the matrix in row-reduced form first.

3.9.1 Examples

1. Find the row-reduced form and the null-space of

main diagonal

$$A = \begin{pmatrix} [illegible] & 2 & 3 \\ 4 & [illegible] & 6 \end{pmatrix}$$

Answer A row-reduced form of the matrix is

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$

Now, some people reserve the term row-reduced (or row-reduced echelon) form for the matrix that also has zeros above the ones. We can get this form in one more step:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

The null space of A can be obtained by solving the system

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So we must have $x_1 = x_3$ and $x_2 = -2x_3$. So the null space is is the line spanned by

$$(1, -2, 1)$$

.

2. Solve the linear system $A\mathbf{x} = \mathbf{y}$ with $\mathbf{y} = (1, 1)$:

   Answer

   Any vector of the form $(z - 1, 1 - 2z, z)$ will do. For instance, $(-1, 1, 0)$.

3. Solve the linear system $A\mathbf{x} = \mathbf{y}$ with $\mathbf{y} = (0, -1)$:

   Answer One example is

$$\left(-\frac{2}{3}, \frac{1}{3}, 0\right)$$

4. Find the row-reduced form and the null space of the matrix

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

   Answer The row-reduced matrix is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

   The null space is spanned by

$$(1, -2, 1)$$

.

5. Find the row-reduced form and the null space of the matrix

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & 1 \end{pmatrix}$$

RowReduce[C]

Answer The row-reduced matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$N(C) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The only element in the null space is the zero vector.

6. Find the null space of the matrix

$$D = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

Answer You can solve the linear system $D\mathbf{x} = \mathbf{y}$ with $\mathbf{y} = (0, 0, 0)$ and discover that $x_1 = -2x_3 = -2x_2$. This means that the null space is spanned $(-2, 1, 1)$. The row-reduced form of the matrix is

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

7. Are the following vectors in R^3 linearly independent or dependent? If they are dependent express one as a linear combination of the others.

NB. ColumRed(A) = RowRed.(A^T)

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix} \right\}$$

Answer The vectors are obviously dependent since you cannot have four linearly independent vectors in a three dimensional space. If you put the matrix in row-reduced form you will get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The first three vectors are indeed linearly independent. Note that the determinant of

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \\ 0 & 3 & 3 \end{pmatrix}$$

is equal to 3.

To find the desired linear combination we need to solve:

$$x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + z \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \\ 0 & 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix}$$

Gaussian elimination could proceed as follows (the sequence of steps is not unique of course): first divide the third row by 3

$$\begin{matrix} 1 & 0 & 1 & 3 \\ 1 & 2 & 2 & 6 \\ 0 & 1 & 1 & 2 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 1 & 3 \\ 0 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 1 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{matrix}$$

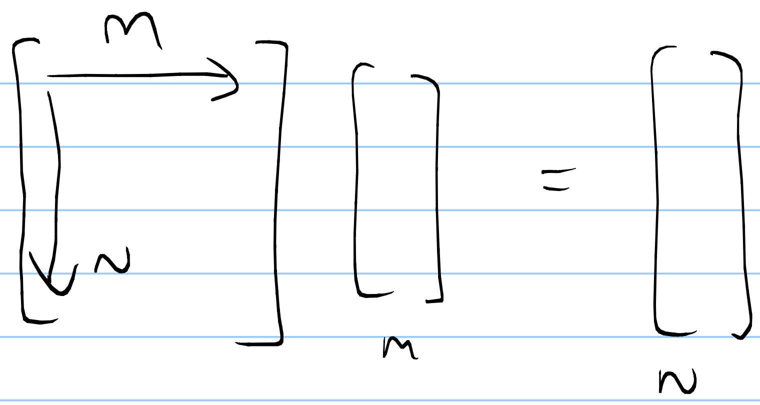
$$\begin{matrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{matrix}$$

Thus we have $z = y = 1$ and $x + z = 3$, which implies that $x = 2$. So, the solution is $(2, 1, 1)$ and you can verify that

$$2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix}$$

3.10 Least Squares

In this section we will consider the problem of solving $A\mathbf{x} = \mathbf{y}$ when no solution exists! I.e., we consider what happens when there is no vector that satisfies the equations exactly. This sort of situation occurs all the time in science and engineering. Often we



Row Space = span(rows)
Column space = Span(columns)
Null space = $N(A)$, $A \cdot x = 0$
Null space of A^T $A^T \cdot x = 0$

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p81 Scales note

A subspace of a vector space is a non empty set S that satisfies

1) the sum of any 2 elements in S is also in S
2) the Scalar multiple of any element in S is also in S.

The test the S is a subspace

$x, y \in S$ and a Scalar α

$\alpha x + y \in S$

E.g. Let $N(A)$ be the set of vectors such that $A \cdot x = 0$

$x, y \in N(A)$

$\underline{\alpha x + y} \in N(A)$

$A \cdot (\alpha x + y) = \alpha \underbrace{A \cdot x}_{=0} + \underbrace{A y}_{=0}$ by assump.

This proves that $N(A)$ is a sub space.

if $A \cdot x = 0$ then $x \in R^m$

Rank: number of linearly IND. rows or columns

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, rank$(A) = 2$

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$ rank$(A) = 2$

Fund. Theorem of Lin. Algebra

$A \in R^{n \times m}$

1) Dim. of the column space = rank = r
2) Dim. of null space $m - r$
3) Dim of row space = r
4) Dim of left null space = $n - r$
   null space (A^T)

$A \cdot x = 0 \Rightarrow N(A) \perp R(A)$

$N(A^T) \perp C(A) = R(A^T)$

colum space

$\underline{Ax = y}$ in order for this system to have a solution y must be in the column space of A

suppose $y \notin C(A)$ not in column space

$\underbrace{Ax - y}_{\perp \ A^T}$ cannot be in col. space

$A^T(Ax - y) = 0$

$$\boxed{A^T A x = A^T y}$$ normal equation

Show later

$$\min_x \|Ax - y\|^2 = A^T A x = A^T y$$

$$\begin{bmatrix} -1 & 1 \\ 1 & 2 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$A^T y$

$$\begin{pmatrix} -1 & 1 & 4 \\ 1 & 2 & 8 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 18 \end{pmatrix}$$

column space

is RHS in column space

$$A^T = \underbrace{\begin{bmatrix} -1 & 1 & 4 \\ 1 & 2 & 8 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & 2 \\ 4 & 8 \end{bmatrix}}_{A} = \begin{bmatrix} 18 & 33 \\ 33 & 69 \end{bmatrix}$$

$$\begin{pmatrix} 18 & 33 \\ 33 & 69 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 18 \end{pmatrix}$$

$$18x + 33y = 9$$
$$33x + 69y = 18$$

$$x = \frac{9 - 33y}{18}$$

$$33\left(\frac{9-33y}{18}\right) + 69y = 18$$

$$\boxed{\begin{array}{l} y = \ldots \\ x \end{array}}$$

$$Ax = y$$

$$\boxed{x = A^{-1}y}$$ in practice this almost never work

1) if A is rectangular $\cancel{A}^{-1}$

2) A^{-1} may not exist even if A is square
Det(A) = 0 or could be very small

we almost always consider

$$\min \underbrace{\|Ax - y\|^2}$$

$$(x, x) = \|x\|^2$$