MATH348 - March 9, 2012
Exam I - 50 Points

NAME: Key#1

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (10 Points)

   (a) True/False: No Justification Needed

   i. If a periodic function has a jump discontinuity then the graph of its Fourier series will display Gibbs' phenomenon.
   False only when truncated

   ii. The Fourier transform of a function with no symmetry will have no symmetry.
   True

   iii. If a function has a Fourier series representation then it has a Fourier transform.
   True

   iv. If f is odd and g is even then $\int_{-1}^{2} f(x)g(x)dx = 0$.
   False

   v. The periodic extension of a function is unique.
   False

   (b) Short Response

   i. A Fourier series is the sum of frequency dependent sinusoids, each of which is multiplied by an amplitude of oscillation. Provide a physical interpretation of each of the underlined terms. Also, provide a deficiency/limitation of Fourier series and a way to resolve this issue.

   Sum: linear combination of waves producing an interference pattern

   Sinusoids: Waves in ~~acustic~~ acoustics these would be pure tones in E+M they would be "colors".

   Amp: ~~Heig~~ Height of wave. Acoustics this is loudness + E+M this is ~~intest~~ intensity.

   F.S. works only for periodic data if we take the width of the period to go to ∞ then we Resolve this w/ F.T.

   ii. What is Gibb's phenomenon and when is it expected to occur?

   Gibb's is an oscillatory Error that occurs ~~when a F.S. i~~ near a jump discontinuity when a F.S. is truncated.

2. (10 Points) Quick Answer Questions

(a) Evaluate $\mathcal{F}\{\delta_{-2}(x)+\delta_2(x)\}$.

$$\mathcal{F}\{\delta_{-2}+\delta_2\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[\delta_{-2}(x)+\delta_2(x)]e^{-i\omega x}\,dx =$$

$$= \frac{1}{\sqrt{2\pi}}\left[e^{2i\omega}\int_{-\infty}^{\infty}\delta_{-2}(x)\,dx + e^{-2i\omega}\int_{-\infty}^{\infty}\delta_2(x)\,dx\right] = \frac{2\cos(2\omega)}{\sqrt{2\pi}}$$

(b) Given the following integrals,

$$\int_{-\pi}^{\pi} f(x)dx = \pi,$$

$$\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \left[\frac{\cancel{\sin(nx)}}{n}\bigg|_{-\pi}^{0} + \frac{\cancel{\sin(nx)}}{n}\bigg|_{0}^{\pi}\right], \quad \int_{-\pi}^{\pi} f(x)\sin(nx)dx = \left[\frac{\cancel{\cos(nx)}}{n}\bigg|_{-\pi}^{0} - \frac{\cancel{\cos(nx)}}{n}\bigg|_{0}^{\pi}\right] = \frac{1-(-1)^n}{n} \quad \frac{(-1)^n-1}{n} = \frac{2-2(-1)^n}{n}$$

$$i\frac{(-1)^n}{n} = \int_{-\pi}^{\pi} g(x)e^{-inx}dx, \quad \frac{e^{in\pi}-e^{-in\pi}}{4\pi} = \int_{-\pi}^{\pi} g(x)dx = 0$$

fill out the following table using yes/no and justify your choices.

	Is Even	Is Odd
$f(x)$	no	no
$g(x)$	no	Yes

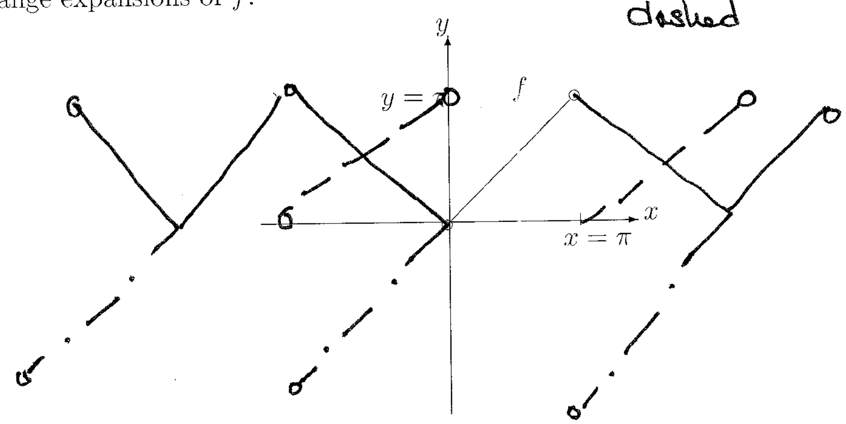
By the previous integrals f has both odd + Even components $\Rightarrow$ no symmetry

$c_n = a_n \pm i b_n$

Since c_n is imaginary and $c_0 = 0$ $g(x)$ is odd.

(c) The graph of f is given below. Label and plot the periodic extension as well as the cosine and sine half-range expansions of f.

dashed Solid dash dot



3. (10 Points) Find the cosine **and** sine half-range expansions of $f(x) = 1$ for $x \in (0, \pi)$.

<u>Cosine:</u> $b_n = 0$, $a_0 = \cancel{\frac{1}{2L}}\ \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{\pi}\int_0^\pi dx = \frac{\pi}{\pi} = 1$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(nx)\,dx = \frac{2}{\pi}\,\frac{\sin(nx)}{n}\Big|_0^\pi = 0 \Rightarrow f_c(x) = 1$$

<u>Sine:</u> $a_0 = a_n = 0$

$$b_n = \frac{2}{L}\int_0^L \sin(nx)\,dx = \frac{-2}{n\pi}\cos(nx)\Big|_0^\pi = \frac{2}{n\pi}\left(1-(-1)^n\right) \Rightarrow$$

$$\Rightarrow f_s(x) = \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi}\sin(nx)$$

4. (10 Points) Find the complex Fourier series representation of $f(x) = x$ for $x \in (-1, 1)$ such that $f(x+2) = f(x)$. Find the complex Fourier series, find the real Fourier series.

$$C_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi}{L}x}\,dx = \frac{1}{2}\int_{-1}^{1} x e^{-i\frac{n\pi}{L}x}\,dx =$$

u	dv
x	$e^{-in\pi/L\,x}$
1	$e^{-in\pi/L\,x} / \left(-\frac{in\pi}{L}\right)$
0	$e^{-i\frac{n\pi}{L}x} / \left(\frac{in\pi}{L}\right)^2$

$$= \frac{1}{2}\left[-\frac{e^{-in\pi}}{in\pi} - \frac{e^{in\pi}}{in\pi}\right] = \frac{i(-1)^n}{n\pi},\ n \neq 0$$

$$C_0 = \frac{1}{2L}\int_{-L}^{L} x\,dx = 0$$

$$f(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i(-1)^n}{n\pi}e^{in\pi x} = \sum_{n=1}^{\infty} \frac{i(-1)^n}{n\pi}e^{in\pi x} - \frac{i(-1)^n}{n\pi}e^{-in\pi x} =$$

$$= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi}\sin(n\pi x)$$

5. (10 Points)

(a) Find the Fourier transform of $f(x) = \begin{cases} 1-|x|, & x \in (-1,1) \\ 0, & x \notin (-1,1) \end{cases}$.

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overset{f(x)}{\cancel{(1-|x|)}}\, e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)\,e^{-i\omega x}\,dx =$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{0}(1+x)\,e^{-i\omega x}\,dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{1}(1-x)\,e^{-i\omega x}\,dx =$$

u	dv
$1 \pm x$	$e^{-i\omega x}$
± 1	$e^{-i\omega x}/-i\omega$
0	$-e^{-i\omega x}/\omega^2$

$$= \frac{1}{\sqrt{2\pi}}\left[\cancel{-\frac{1}{i\omega}} + \frac{1}{\omega^2} - \frac{e^{i\omega}}{\omega^2} + \cancel{\frac{1}{i\omega}} - \frac{e^{-i\omega}}{\omega^2} + \frac{1}{\omega^2}\right] =$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2-2\cos(\omega)}{\omega^2}\right]$$

(b) Evaluate $\mathcal{F}\{f(x+1)\}$ $= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x+1)\,e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{-i\omega(u+\cancel{1})}\,du =$

$$= e^{-i}\,\mathcal{F}\{f\}$$

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NAME: Key 2

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1. (10 Points)
   (a) True/False: No Justification Needed
      i. A truncated Fourier sine half-range expansion of a continuous function can have Gibb's phenomenon. False
      ii. If a function is even then its Fourier transform will be odd.
      False
      iii. If a function has a Fourier transform then it has a Fourier series.
      False
      iv. If f is odd and g is odd then $\int_{-2}^{2} \underbrace{f(x)g(x)}_{\text{Even}}dx = 0$.
      False
      v. A function can have many periodic extensions.
      True
   (b) Short Response
      i. A Fourier series is the <u>sum</u> of frequency dependent <u>sinusoids</u>, each of which is multiplied by an <u>amplitude of oscillation</u>. Provide a physical interpretation of each of the underlined terms. Also, provide a deficiency/limitation of Fourier series and a way to resolve this issue.

      See Key 1

      ii. What is a periodic extension, when and why can it be used?

      If a fn is "Reasonable" and given on a finite domain then this can be repeated into the unused x space ⇒ periodic ⇒ F.S. Rep of original fn

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2. (10 Points) Quick Answer Questions

(a) Evaluate $\mathcal{F}\{\delta_{2}(x) - \delta_{-2}(x)\}$. See Key one and note *

*

$$\Rightarrow \mathcal{F}\{\quad\} = \frac{1}{\sqrt{2\pi}}\left[e^{-2i\omega} - e^{2i\omega}\right] = \frac{-2i}{\sqrt{2\pi}}\sin(2\omega)$$

(b) Given the following integrals,

$$\int_{-\pi}^{\pi} f(x)dx = \pi,$$

$$\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \left[\frac{\sin(nx)}{n}\Big|_{-\pi}^{0} + \frac{\sin(nx)}{n}\Big|_{0}^{\pi}\right], \quad \int_{-\pi}^{\pi} f(x)\sin(nx)dx = \left[\frac{\cos(nx)}{n}\Big|_{-\pi}^{0} - \frac{\cos(nx)}{n}\Big|_{0}^{\pi}\right],$$

$$i\frac{(-1)^n}{n} = \int_{-\pi}^{\pi} g(x)e^{-inx}dx, \quad \frac{e^{in\pi} - e^{-in\pi}}{4\pi} = \int_{-\pi}^{\pi} g(x)dx.$$

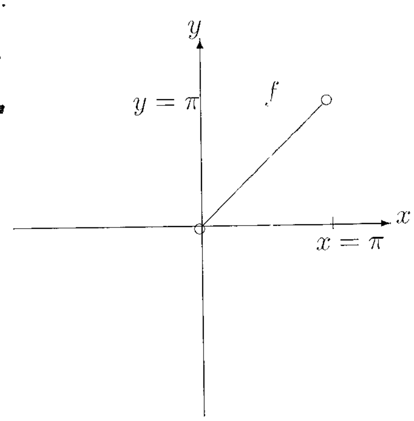
fill out the following table using yes/no and justify your choices.

	Is Even	Is Odd
$f(x)$		
$g(x)$		

See Key #1

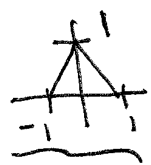
(c) The graph of f is given below. Label and plot the periodic extension as well as the cosine and sine half-range expansions of f.

See Key #1



3. (10 Points) Find the cosine **and** sine half-range expansions of $f(x) = 1$ for $x \in (0, \pi)$.

See Key 1



4. (10 Points) Find the complex Fourier series representation of $f(x) = 1 - |x|$ for $x \in (-1, 1)$ such that $f(x+2) = f(x)$. From the complex Fourier series, find the real Fourier series.

See Key #1 Prob 5 for integration and note

$$w = w_n = \frac{n\pi}{L} = n\pi, \quad \text{where } L = 1$$

thus

$$C_n = \frac{1}{2}\left[\frac{2 - 2\cos(n\pi)}{n^2\pi^2}\right] = \frac{1-(-1)^n}{n^2\pi^2}, \quad n \neq 0$$

$$C_0 = \frac{1}{2}\int_{-1}^{1}(1-|x|)\,dx = \frac{1}{2}\cdot\frac{1}{2}\cdot 2\cdot 1 = \frac{1}{2}$$

$$\Rightarrow f(x) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi^2}\right)e^{in\pi x} = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2\pi^2}\Big(\underbrace{e^{in\pi x} + e^{-in\pi x}}_{2\cos(n\pi x)}\Big)$$

5. (10 Points)

(a) Find the Fourier transform of $f(x) = \begin{cases} 1, & x \in [0,1) \\ -1, & x \in (-1,0) \\ 0, & x \notin (-1,1) \end{cases}$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{0} -e^{-i\omega x} dx + \int_{0}^{1} e^{-i\omega x} dx \right] =$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\omega x}}{i\omega} \Big|_{-1}^{0} - \frac{e^{-i\omega x}}{i\omega} \Big|_{0}^{1} \right] = \frac{1}{i\omega} - \frac{e^{i\omega}}{i\omega} - \frac{e^{-i\omega}}{i\omega} + \frac{1}{i\omega} =$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2}{i\omega} - \frac{2\cos(\omega)}{i\omega} \right]$$

(b) Evaluate $\mathcal{F}\{f(3x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(3x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-i\omega \frac{u}{3}} \frac{du}{3} =$

$$= \frac{1}{3} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-ivu} du = \frac{1}{3} \hat{f}(v) = \frac{1}{3} \hat{f}\left(\frac{\omega}{3}\right)$$

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NAME: Key 3

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1. (10 Points)

   (a) True/False: No Justification Needed

   i. If a periodic function is neither even nor odd then its Fourier series representation must have sine terms/modes. True

   ii. The Fourier transform of an odd function is odd.
   True

   iii. There are real Fourier series that cannot be represented with complex Fourier series.
   False

   iv. If f is odd and g is even then $\int_{-1}^{2} f(x)g(x)dx = 0$.
   False

   v. The function e^{ix} has odd symmetry.
   False

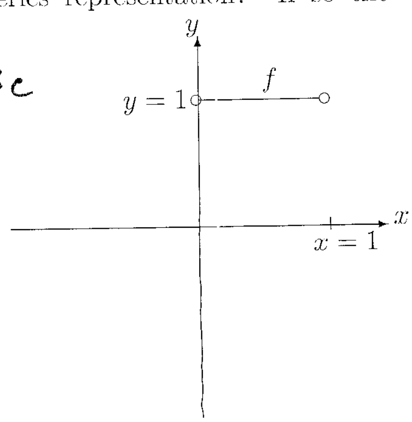
   (b) Short Response

   i. A Fourier series is the sum of frequency dependent sinusoids, each of which is multiplied by an amplitude of oscillation. Provide a physical interpretation of each of the underlined terms. Also, provide a deficiency/limitation of Fourier series and a way to resolve this issue.

   See Key #1

   ii. Does f have a Fourier series representation? If so the will it contain any cosine functions?

   Yes by periodic Extension.



   No, see Key #1 regarding Problem 3 on ~~this~~ that Exam.

2. (10 Points) Quick Answer Questions

(a) Evaluate $\mathcal{F}\{\delta_{-2}(x) - \delta_2(x)\}$.

See Key #1, #2 to get

$$\mathcal{F}\{\quad\} = \frac{2i\sin(2w)}{\sqrt{2\pi}}$$

(b) Given the following integrals,

$$\int_{-\pi}^{\pi} f(x)dx = \pi,$$

$$\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \left[\frac{\sin(nx)}{n}\bigg|_{-\pi}^{0} + \frac{\sin(nx)}{n}\bigg|_{0}^{\pi}\right], \quad \int_{-\pi}^{\pi} f(x)\sin(nx)dx = \left[\frac{\cos(nx)}{n}\bigg|_{-\pi}^{0} - \frac{\cos(nx)}{n}\bigg|_{0}^{\pi}\right],$$

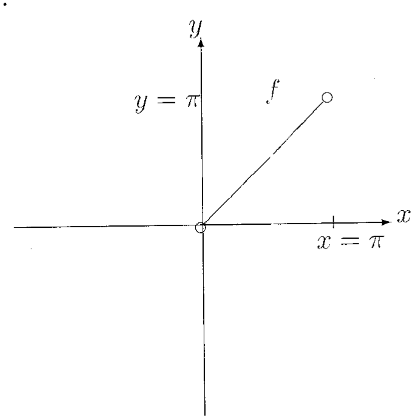
$$i\frac{(-1)^n}{n} = \int_{-\pi}^{\pi} g(x)e^{-inx}dx, \quad \frac{e^{in\pi} - e^{-in\pi}}{4\pi} = \int_{-\pi}^{\pi} g(x)dx.$$

fill out the following table using yes/no and justify your choices.

	Is Even	Is Odd
$f(x)$		
$g(x)$		

Key #1

(c) The graph of f is given below. Label and plot the periodic extension as well as the cosine and sine half-range expansions of f.



Key #1

3. (10 Points) Find the cosine **and** sine half-range expansions of $f(x) = x$ for $x \in (0, \pi)$.

u	dv_1	dv_2
x	$\cos(nx)$	$\sin(nx)$
1	$+\frac{\sin(nx)}{n}$	$-\frac{\cos(nx)}{n}$
0	$-\frac{\cos(nx)}{n^2}$	$-\frac{\sin(nx)}{n^2}$

Cosine: $b_n = 0$,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_c(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\,\frac{1}{2}x^2\Big|_0^{\pi} = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

$$\Rightarrow f_c(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]\cos(nx)$$

Sine: $a_0 = a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cancel{f(x)}\,x\,\sin(nx)\,dx = \frac{2}{\pi}\left[\frac{-\pi\cos(n\pi)}{n}\right] = \frac{2(-1)^{n+1}}{n} \Rightarrow$$

$$f_s(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n^2}\sin($$

4. (10 Points) Find the complex Fourier series representation of $f(x) = \begin{cases} 1, & x \in (0,1) \\ -1, & x \in (-1,0) \end{cases}$, $f(x+2) = f(x)$.
From the complex Fourier series, find the real Fourier series.

See the integration from Key #2 problem 5 and note

$c_0 = 0$ since f is odd

and

$$\omega = \omega_n = \frac{n\pi}{L} = n\pi, \quad L = 1$$

to get

$$c_n = \frac{i}{2}\left[\frac{2\cos(n\pi)}{n\pi} - \frac{2}{n\pi}\right] = \frac{i}{n\pi}\left[(-1)^n - 1\right]$$

thus,

$$f(x) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{i\left[(-1)^n - 1\right]}{n\pi} e^{+in\pi x} = \sum_{n=1}^{\infty} \frac{i\left[(-1)^n - 1\right]}{n\pi}\left(e^{in\pi x} - e^{-in\pi x}\right)$$

$$= \sum_{n=1}^{\infty} \frac{2\left[1 - (-1)^n\right]}{n\pi}\sin(n\pi x)$$

5. (10 Points)

(a) Find the Fourier transform of $f(x) = \begin{cases} x, & x \in (-\pi, \pi) \\ 0, & x \notin (-\pi, \pi) \end{cases}$

u	dv
x	$e^{-i\omega x}$
1	$e^{-i\omega x}/-i\omega$
0	$-e^{-i\omega x}/\omega^2$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x e^{-i\omega x}\, dx =$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{x e^{-i\omega x}}{-i\omega} \Big|_{-\pi}^{\pi} + \frac{e^{-i\omega x}}{\omega^2} \Big|_{-\pi}^{\pi} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{-\pi}{i\omega} \left(e^{-i\omega\pi} + e^{i\omega\pi} \right) + \frac{e^{-i\omega\pi}}{\omega^2} - \frac{e^{i\omega\pi}}{\omega^2} \right] =$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\frac{2\pi \cos(\omega\pi)}{i\omega} - \frac{2i \sin(\omega\pi)}{\omega^2} \right]$$

(b) Assuming that $\hat{f}$ exists, evaluate $\mathcal{F}\{f'(x)\}$.

$u = e^{-i\omega x}$ $\quad dv = f'\,dx$

$du = -i\omega e^{-i\omega x}\,dx$ $\quad v = f$

$$\mathcal{F}\{f'\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-i\omega x}\, dx =$$

$$= \frac{1}{\sqrt{2\pi}} \left[\underbrace{f(x) e^{-i\omega x} \Big|_{-\infty}^{\infty}}_{=0 \text{ since } \lim_{x \to \pm\infty} f = 0} + i\omega \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx \right] = i\omega \mathcal{F}\{f\}$$