MATH348: COMPLEX FOURIER SERIES

Those who dream by day are cognizant of many things that escape those who dream only at night.

1. Introduction

It is known that every vector space has a basis[1] and simple to show that such bases are not unique. Regardless, it should not surprise the reader that there are other choices for the bases elements of a Fourier series. Why? Well, any Fourier mode can be thought of as a solution to the following simple harmonic oscillator,

$$my'' + ky = 0, \tag{1}$$

for a specific length-scale set by the frequency $\omega = \sqrt{k/m}$. In the case of a mass-spring system, this is a temporal-frequency. However, if we think of this as angular-frequency of a spatial wave the we write the general solution as,

$$y(x) = a_1 \cos(\omega x) + b_1 \sin(\omega x). \tag{2}$$

If we are given infinitely-many oscillator equations, indexed by $n = 0, 1, 2, \ldots$ and we summed them up, as we would as a subpart of separation of variables, then we might get something like

$$f(x) = \sum_{n=0}^{\infty} y_n(x) = \sum_{n=0}^{\infty} a_n \cos(\omega_n x) + b_n \sin(\omega_n x), \tag{3}$$

and if ω_n appear as multiples of a base angular frequency, $\omega_n = n\pi/L$, then we have a Fourier series. [2] However, it is likely that the sine/cosine solution to the SHO was not the first one shown to you. What probably happened was writing down solutions like,

$$y(x) = c_1 e^{i\omega x} + c_{-1} e^{-i\omega x}, \tag{4}$$

and manipulating them into the sine/cosine form via Euler's formula. Thus, it shouldn't surprise you that if a Fourier series as a real sine/cosine form then it

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[1]The statement is trivial for finite-dimensional spaces like $\mathbb{R}^n$, where we continue to build dimensions based on our intuition of $\mathbb{R}$. However, in the infinite-dimensional case the problem is much more complicated. In fact, it is equivalent to the axiom of choice, which is a statement that is independent of mathematical theory at large. That is, it is impossible to prove that the statement, *every vector space has a basis* is true **and** it is impossible to prove that the statement is false. This is maybe too abstract. A simpler example is Euclid's parallel line postulate, which states that two parallel lines will never intersect. This statement is independent of geometry and can be taken to be either true or false. If we are in a flat space then it is decidedly true. If, however, we were in a curved space then it is false. If you and your friend started walking north, parallel to each other, then eventually you would meet at a pole.

[2]Normally, we start the sum at 1 but here we are starting at 0. Notice that if $\omega_n = n\pi/L$ then $\cos(\omega_0 x) = 1$ and $\sin(\omega_0 x) = 0$ and we are back to the old notation. I will almost never write it this way because the coefficient definitions for the zeroth-terms takes on a slightly different form.

should also have a imaginary exponential form. The following will explore the complex form of the Fourier series.

2. Complex Fourier Series

When working with complex functions/vectors, the inner-product of the space[3] of *reasonable*[4] $2L$–periodic functions takes the form,

$$\langle f, g\rangle = \int_a^b f(x)\bar{g(x)}dx, \quad b - a = 2L > 0, \tag{5}$$

where $\bar{z} = z^*$ denotes complex conjugation. That said, the following forms an alternate basis for this space,

$$\mathcal{B} = \left\{\dots, e^{-i\omega_3 x}, e^{-i\omega_2 x}, e^{-i\omega_1 x}, 1, e^{i\omega_1 x}, e^{i\omega_2 x}, e^{i\omega_3 x}, \dots\right\} \tag{6}$$

and thus for any $f \in \mathcal{P}_{2L}$ there exists complex-valued coefficients, $c_n \in \mathbb{C}$, such that

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x}. \tag{7}$$

It can be shown that these basis elements obey the orthogonality relation,

$$\left\langle e^{i\omega_n x}, e^{-i\omega_m x}\right\rangle = 2L\delta_{nm}, \tag{8}$$

which then allows us to write down the complex Fourier coefficient formula,

$$c_n = c(\omega_n) = \frac{1}{2L}\left\langle f, e^{i\omega_n x}\right\rangle. \tag{9}$$

It must be stressed that every Fourier series of the form, Eq. (3), has an equivalent representation as Eq. (7) with coefficients Eq. (9).

For example beginning with our old sawtooth wave, $f(x) = x$, where $x \in (-\pi, \pi)$ such that $f(x + 2\pi) = f(x)$, we find

$$c_n = \frac{i(-1)^n}{n}, \quad n \neq 0, \tag{10}$$

and $c_0 = 0$. From this and the complex Fourier series we get,

$$f(x) = \sum_{n=-\infty}^{-1} c_n e^{i\omega_n x} + \sum_{n=1}^{\infty} c_n e^{i\omega_n x} \tag{11}$$

$$= \sum_{n=1}^{\infty} c_n e^{i\omega_n x} + c_{-n} e^{-i\omega_n x} \tag{12}$$

$$= \sum_{n=1}^{\infty} \frac{i(-1)^n}{n}(e^{i\omega_n x} - e^{-i\omega_n x}) \tag{13}$$

$$= \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nx), \tag{14}$$

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[3]An inner-product space is a vector-space whose elements obey a product on vectors that returns a scalar and is *symmetric, linear and positive semi-definite.*

[4]Note that we have yet to define what reasonable means. One could take this to mean that the function is good enough so that the integrals in the Fourier coefficient formulae are finite. However, since we have not addressed the convergence of the series, the situation is more complicated and a matter of formal analysis.

which is the same result as before. So, why even use this if it the same as the real case? Well, even though they do not explicitly show physical symmetries, they are in an exceptionally compact. This can make certain analyses easier.

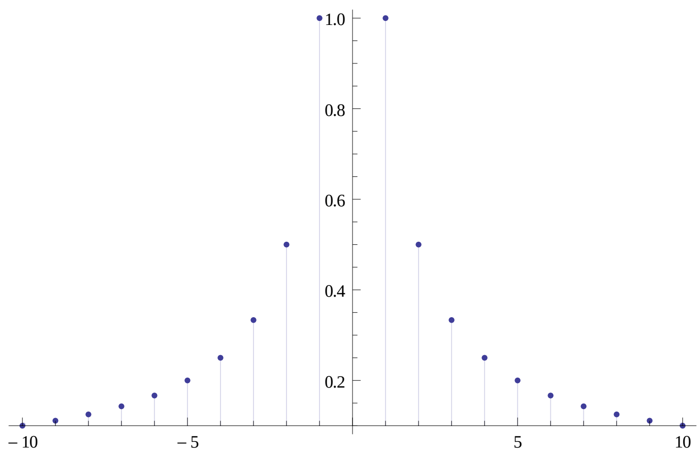
First, let's go back to the SHO and notice the general solution takes the form of Eq. (4) which can be thought of is as single mode of a Fourier series.[5] Also, remember that solutions to the SHO must also obey conservation of mechanical energy,

$$\frac{m\dot{y}^2}{2} + \frac{ky^2}{2} = E \in [0, \infty). \tag{15}$$

It is easier to substitute the complex form of the solution into the previous equation to get

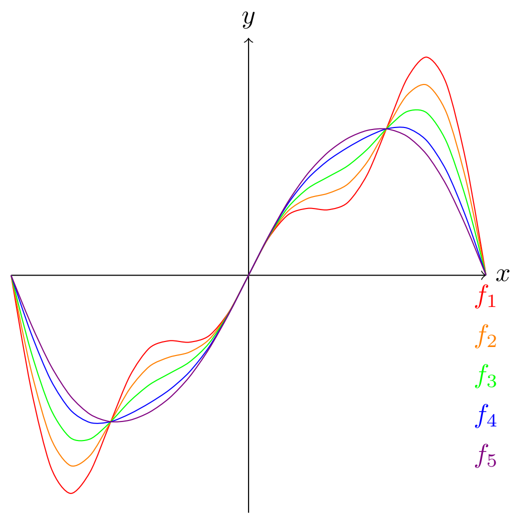
$$2kc_1c_{-1} = 2kc_1\bar{c_1} = 2k|c_1|^2 = 2k(a_1^2 + b_1^2) = E, \tag{16}$$

which says that the energy of the mode is proportional to the square of the amplitude of the wave. Thus the mechanical energy in a single-cycle of the sawtooth wave obeys $E \propto \sum_{n=1}^{\infty} \frac{1}{n^2}$. Maybe what is more important is that we get another perspective on Fourier series and that is the concept of a power-spectrum. That is, if we graph $|c_n|^2 = |c(\omega_n)|^2$ against frequency, ω we get the following:



This graph tells us that the original signal is made up of many frequency components that are separated an equidistant amount in frequency space. Moreover, the "energy" needed for each mode is finite and decreasing in the ω–variable. So, we have two perspectives of the saw-tooth function. The first, is the graph of amplitude versus space or time and the second is "energy" versus frequency. This duality is at the heart of Fourier and signal analysis. For instance, if you wanted to artificially increase the bass-tones in the original signal, you could amplify all c_n associated with ω_n in the bass range. This is a naive way to think of a low-pass filter, which actually does nothing to the low-frequency components but attenuates high-frequency components. The following is a graph showing the affect of changing the Fourier coefficients of the sawtooth modes.

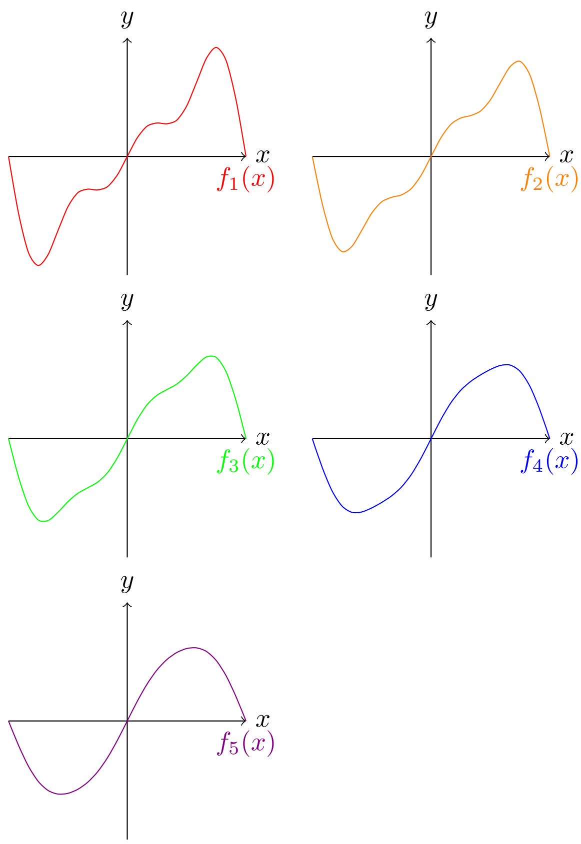
[5] Here we use the term mode to indicate those functions associated with an angular frequency, both positive or negative in the case of the complex notation.



Specifically, we have used the following definitions:

$$f_1(x) = 2\sin(x) - \sin(2x) + \frac{2}{3}\sin(3x),$$
$$f_2(x) = 2\sin(x) - \frac{3}{4}\left(\sin(2x) - \frac{2}{3}\sin(3x)\right),$$
$$f_3(x) = 2\sin(x) - \frac{1}{2}\left(\sin(2x) - \frac{2}{3}\sin(3x)\right),$$
$$f_4(x) = 2\sin(x) - \frac{1}{4}\left(\sin(2x) - \frac{2}{3}\sin(3x)\right),$$
$$f_5(x) = 2\sin(x) - \frac{1}{8}\left(\sin(2x) - \frac{2}{3}\sin(3x)\right).$$

To make the affect easier to see, we plot each graph individually.



The idea here is that the higher-frequency modes are responsible for building the function $y(x) = x$, where $x \in (-\pi, \pi)$ such that $f(x + 2\pi) = f(x)$, from the fundamental mode $\sin(x)$. In these graphs we dampen the affect of these modes until we achieve a function, f_5, that is like the fundamental mode but does have non-trivial higher-frequency components.

The concept of a Fourier series is a deep one that has connections to many other mathematical and physical concepts. At this point we can say that a Fourier series is a standard way to decompose a function, defined on a closed and bounded portion of the real-line, into sinusoids whose frequencies are multiples of a common fundamental/base frequency. Since the mechanical energy of a wave is proportional to the square of its amplitude, the Fourier decomposition gives us access to the single-cycle energy of the linear combination sinusoids. Maybe the most important quality of a Fourier representation is that it allows us to think of the function in two ways, time-space and frequency-space. The next thing to address is what happens when the spacing of energy in frequency space is not equal.

3. Things to Do

1. Show that Eq. (5) defines satisfies the rules of inner-product.
2. Show that the elements in $\mathcal{B}$ satisfy Eq. (8).
3. Using Eq. (7)-(8) derive Eq. (9).
4. If you did not follow the derivation of Eq. (10) from class then you should re-do it.

5. If you did not follow the derivation of Eq. (14) from the complex Fourier series representation of the sawtooth function then you should re-do it.
6. Find the complex Fourier series representation of $f(x) = x^2$ where $x \in (-\pi, \pi)$ such that $f(x + 2\pi) = f(x)$.
7. From the complex Fourier series representation of $f(x) = x^2$, where $x \in (-\pi, \pi)$ such that $f(x + 2\pi) = f(x)$, find the corresponding real Fourier series.

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