9/19/12

The heat Eqn:

The Equation

$$U_t = c^2 U_{xx}, \quad (\mathrm{I})$$

is called the 1D Homogeneous heat or diffusion Eqn. It is called heat b/c it was one of the first PDE studied by Fourier. However, it is a general Eqn that Evolves a density u which is controlled by the second-law of thermodynamics.

Step 1: $U(x,t) = X(x)T(t) \Rightarrow$ for $X(x) \neq 0$, $T(t) \neq 0$

$$(\mathrm{I}) \Leftrightarrow \frac{T'}{c^2 T} = \frac{X''}{X} = -\lambda \in \mathbb{R}$$

$$\Rightarrow T' = -\lambda c^2 T$$

$$X'' + \lambda X = 0$$

Step 2: Recall that the spatial ODE has the soln set

$\lambda > 0: \; X_1(x) = C_1 \sin(\sqrt{\lambda} x) + C_2 \cos(\sqrt{\lambda} x)$

$\lambda < 0: \; X_2(x) = C_3 \sinh(\sqrt{|\lambda|} x) + C_4 \overset{\cosh}{\cancel{\sinh}}(\sqrt{|\lambda|}\, x)$

$\lambda = 0: \; X_3(x) = C_5 x + C_6$

There are two interesting boundary conditions to consider for $x \in (0, L)$

(II) $X(0) = 0, \; X(L) = 0$ [See lecture 9.7.12

(II') $X'(0) = 0, \; X'(L) = 0$ [See lecture 9.19.12

(II) $\Rightarrow X_n(x) = \sin(\sqrt{\lambda_n} x), \; \sqrt{\lambda_n} = \frac{n\pi}{L}, \; n = 1, 2, 3, \ldots$

(II') $\Rightarrow X_n(x) = \cos(\sqrt{\lambda_n} x), \; \sqrt{\lambda_n} = \frac{n\pi}{L}, \; n = 0, 1, 2, 3$

(*) Recall if $n = 0 \Rightarrow X_0(x) = 1$, which is the constant soln for $\lambda = 0$

Now, we ~~see~~ return to time we have

$$T_n' = -\lambda_n c^2 T_n \quad , \; n = \underline{\underline{0}}, 1, 2, \cdots$$

If $X'(0) = 0$
$X'(L) = 0$

which asks what fn T_n produces a $-\lambda_n c^2$ multiple of itself ~~if~~ upon 1 diff. step.

$$T_n(t) = A_n e^{-\lambda_n c^2 t} \quad , \; A_n \in \mathbb{R}, \; n = \underline{\underline{0}}, 1, 2, \cdots$$

Which gives the general sol.

$$\underline{u(x,t)} = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n} x) e^{-c^2 \lambda_n t}$$

or

$$\underline{u(x,t)} = \sum_{n=0}^{\infty} A_n \cos(\sqrt{\lambda_n} x) e^{-c^2 \lambda_n t} = \underline{\underline{A_0}} + \sum_{n=1}^{\infty} A_n \cos(\sqrt{\lambda_n} x) e^{-c^2 \lambda_n t}$$

dep. on (II') or (II).

Key Points:

- $U = U(x,t)$ is a density $[U] = \frac{\text{Stuff}}{\text{length}}$
- Stuff could be:
  - Heat Energy $\rightarrow$ U = temp
  - Mass ~~Density~~ $\rightarrow$ U = density of ~~maybe~~ maybe impurity
  - Probability $\rightarrow$ Prob density
- C^2 is called diffusivity, $[C^2] = \frac{\text{length}^2}{\text{time}}$ and measures how the much/easy stuff ~~want~~ is allowed to flow/s through the object.
- $[\lambda_n C^2] = \frac{1}{\text{length}^2} \cdot \frac{\text{length}^2}{\text{time}} = \frac{1}{\text{time}}$, decay rate.

- If we think about U as temp then:

i) (II) $\Rightarrow$ the object touches a universe of zero temp on Relative scale

ii) (II') $\rightarrow$ the object's temp has zero slope in temp at Edges. $\Rightarrow$

$\Rightarrow$ no local ~~the~~ temp diff $\Rightarrow$ no heat flow b/c of local Equilibrium. } Ideal Insulati

- In these cases:

i) $\lim_{t\to\infty} U(x,t) = 0$, the object attains Equilibrium with universe as $t\to\infty$.

ii) $\lim_{t\to\infty} U(x,t) = A_0 = \frac{1}{L}\int_0^L U(x,0)\,dx =$

$= U_{average}(x,0)$, the object Establishes a Consta Equilibrium state that is the average of the initial temp as $t\to\infty$

- If $c^2 = \frac{K}{\sigma\rho}$, $K \equiv$ thermal conductivity
$\sigma \equiv$ Specific heat
$\rho \equiv$ density

then as K increases for fixed σ, ρ we have faster decay to Equilibrium.

Also, as ρ increases for fixed K, σ we have slower decay to Equilibrium.

- Lastly if $u(x,0) = f(x)$ then

(II) $u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n}\, x)$

$\Rightarrow A_n = \frac{2}{L}\int_0^L f(x) \sin(\sqrt{\lambda_n}\, x)\, dx$

(II') $u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\sqrt{\lambda_n}\, x)$

$\Rightarrow A_0 = \frac{1}{L}\int_0^L f(x)\,dx$, $A_n = \frac{2}{L}\int_0^L f(x) \cos(\sqrt{\lambda_n}\, x)\, dx$

So, if $f(x) = \begin{cases} \frac{2k}{L}x, & x \in (0, \frac{L}{2}) \\ \frac{2k}{L}(L-x), & x \in [\frac{L}{2}, L) \end{cases}$

(II) $A_n = \frac{8k}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$, [See 9.14.12]

(II') $A_0 = \frac{k}{2}$, $A_n = \frac{8k}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right]$ [See Problem 1 from this HW!]

Key Point:

The modes that make the triangle are the same, as they should be, but the time dynamics are not, which is Expected b/c diffusion is different than ideal vibrations.