PHGN 462 Homework 6

1) (From Pollack and Stump 15.2). I like this one a lot. Ever since Phys 200 we've been talking about the field made by an infinite current-carrying wire, but we couldn't talk about the part where we actually turn on the current. Now we can.

Suppose that at t = 0 a current I is suddenly established in an infinite wire that lies on the z axis. Show that the resulting electric and magnetic fields are:

$$E_z(r,t) = \frac{-\mu_0 I c}{2\pi\sqrt{c^2t^2 - r^2}}$$

$$B_\phi(r,t) = \frac{\mu_0 I}{2\pi r}\frac{ct}{\sqrt{c^2t^2 - r^2}}$$

Also show that after a long time, $t \gg r/c$, the magnetic field is the same as the static field of a long wire with constant current I. What is the electric field for $t \gg r/c$?

2) (From Pollack and Stump 15.15)

The average power radiated by an oscillating dipole, with dipole moment $p(t) = p_0\cos(\omega t)$, is

$$P_{avg} = \frac{p_0^2\omega^4}{12\pi\varepsilon_0 c^3}$$

Derive this result from the Larmor formula, treating the dipole as an oscillating pair of charges $\pm q_0$, which oscillate 180 degrees out of phase with amplitude of oscillation *d*/2. (Note that $p_0 = q_0 d$) But be careful! The waves radiated by q_0 and $-q_0$ interfere, so in Larmor's formula you must add the *qa*'s and then square, rather than adding the squares of the *qa*'s!

3) The solutions to the differential equations for V and **A** in the Lorentz gauge (equations 15.5 and 15.6 in Pollack and Stump, or see lecture notes) are pretty clean as long as you evaluate the charge and current densities at the retarded time $t - {}^{r}/_{c}$. This represents the fact that influences from a charge or current travel at some speed c and take an amount of time ${}^{r}/_{c}$ to reach some observation point. None of this is super shocking to most people.

You know what *is* kind of shocking? The retarded potentials for V and **A** (equations 15.19 and 15.20 in Pollack and Stump, or notes) don't only work with $t - {}^{r}/_{c}$. They also work with $t + {}^{r}/_{c}$, called the advanced time. This is an example of the fact that most physical laws are invariant with respect to time reversal – that is, *for the most part*, nature doesn't care which direction time flows; the laws of nature still work in either direction.

And since at some point I'm supposed to ask you to actually do something: Prove that 15.19 and 15.20 are, in fact, still solutions to 15.5 and 15.6 even with the advanced time instead of the retarded time. And by that I mean prove that 15.19 solves 15.5, since doing 15.20 for 15.6 is the exact same process on a component by component basis.

4) (Read *all* of the below)

First do Pollack and Stump 15.3. This innocent-looking problem actually puts a nice happy bow on all of classical electromagnetism, and it's fairly short:

Starting from Maxwell's equations with sources $\rho(\vec{x},t)$ and $\vec{J}(\vec{x},t)$ in vacuum, show that:

$$-\nabla^2\vec{E} + \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = -\frac{1}{\varepsilon_0}\nabla\rho - \mu_0\frac{\partial\vec{J}}{\partial t}$$

$$-\nabla^2\vec{B} + \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = \mu_0\nabla\times\vec{J}$$

Now let's expand on that. Maxwell's equations in differential form,

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{E} = \frac{-\partial\vec{B}}{\partial t} \qquad \nabla\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

sometimes cause people to conclude odd things because they treat changing electric and magnetic fields as sources on an equal footing with charges and currents. But of course, those aren't *really* equal. Ultimately the changing electric and magnetic fields themselves came from charges and currents. Indeed, eventually all E and B fields come from charges and currents, period. The equations that this problem had you derive make that clearer: They're differential equations for E & B that explicitly lay out charges and currents as the ultimate sources of the fields.

If you ever take graduate E&M, you may have the good fortune to solve these differential equations using Green's functions to get the following integral solutions:

$$\vec{E}(x,t) = \frac{1}{4\pi\varepsilon_0}\int\left\{\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\rho(\vec{x}',t') + \frac{\vec{x}-\vec{x}'}{c|\vec{x}-\vec{x}'|^2}\frac{\partial\rho(\vec{x}',t')}{\partial t'} - \frac{1}{c^2|\vec{x}-\vec{x}'|}\frac{\partial\vec{J}(\vec{x}',t')}{\partial t'}\right\}d^3x'$$

$$\vec{B}(x,t) = \frac{\mu_0}{4\pi}\int\left\{\vec{J}(\vec{x}',t')\times\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} + \frac{\partial\vec{J}(\vec{x}',t')}{\partial t'}\times\frac{\vec{x}-\vec{x}'}{c|\vec{x}-\vec{x}'|^2}\right\}d^3x'$$

with everything in the integrand evaluated at the retarded time, as usual. These are known as Jefimenko's equations, and are the complete integral prescription for figuring out fields, given complete information about the charges and currents in the neighborhood. Hopefully you're still reading, because this is the place where I tell you that part of the credit for this problem comes from you commenting on how awesome this Jefimenko perspective is. Also examine the structure of the Jefimenko equations and comment on whether they make any intuitive sense and why (or why not, if that's how you feel).

5) (From Pollack and Stump 10.34)

The differential equation for a harmonically driven series LRC circuit is

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = \varepsilon_0 \sin(\omega t)$$

a) Let L = 1, C = 1, and R = 0.1, in some system of units. Solve the equation numerically, for initial values $Q(0) = 0$ and $I(0) = 0$, and driving frequency $\omega = 0.5$. After a transient period, the solution settles down to a steady state oscillation at the driving frequency ω.

b) Now vary ω, in the range from 0.5 to 1.5. Show, by plots of $Q(t)$ and $I(t)$, that the steady state oscillations have maximum amplitude if $\omega \approx 1/\sqrt{LC}$. Explain this result in terms of resonance.

c) Show graphically that there is a phase shift between the applied EMF and the charge on C.

d) Also explain how to come up with that given differential equation, and derive the analytic steady-state solution for $Q(t)$, either with Mathematica (show the code), or by hand. Write it in terms of ω_0, the resonance frequency for an LC or RLC circuit. As an easy check, you can make sure your solution reproduces eqn. 10.33 in the book, $Q(t) = \frac{\varepsilon_0}{L}\frac{\cos(\omega t)}{\omega_0^2 - \omega^2}$, when R goes to zero.