MATH348 - April 11, 2011 Exam II - 50 Points	NAME:

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (10 Points) Conceptual Questions. For the following questions assume that we are considering the physical problem on a bounded domain, $x \in [0, 1]$.

   (a) Write down the heat and wave equations and also any initial conditions needed for unique solutions.

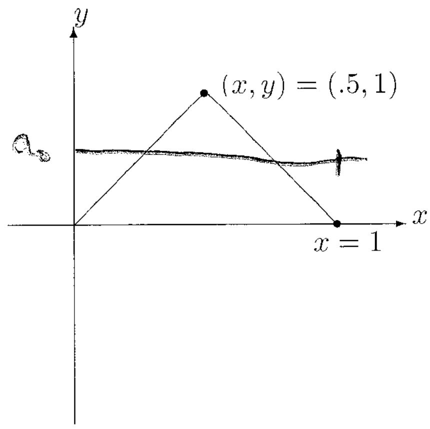
   $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ; \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

   $u(x,0) = f(x)$ $\qquad$ $u(x,0) = f(x)$, $u_t(x,0) = g(x)$

   (b) Assume that the following graph is the initial temperature for a homogeneous heat problem with boundary conditions, $u_x(0, t) = 0$, $u_x(1, t) = 0$. Describe the physical meaning of these boundary conditions and graph the temperature profile for $t \to \infty$.

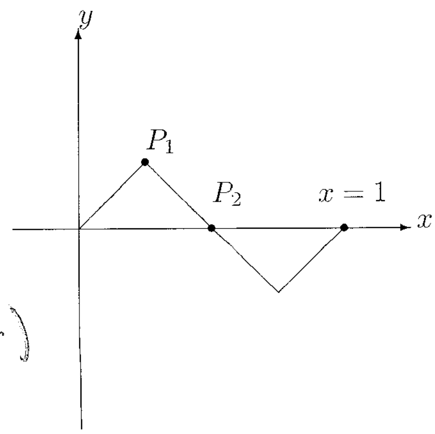
   Insulation (Ideal) at both end pts.

   $\frac{1}{2} = \frac{1}{1}\int_0^L f\,dx = a_0$



   (c) Assume that the following graph is the initial displacement of an elastic string modeled by the homogeneous wave equation subject to boundary conditions $u(0, t) = 0$, $u(1, t) = 0$. Describe the physical meaning of these boundary conditions and describe time-dynamics of the points, P_1 and P_2, on this elastic string assuming that the string has no initial velocity.

   Fixed at both ends.



   P_1 oscillates

   P_2 static (Node pt)

2. (10 Points) Quick Problems

(a) Apply separation of variables to $u_t = u_{xx} + u_{yy}$ and find three ordinary differential equations consistent with the PDE.

Assume $u(x,y,t) = G(t)\,F(x,y)$

$$\Rightarrow \quad u_t = G'F = F_{xx} + F_{yy} = u_{xx} + u_{yy}$$

$$\Leftrightarrow \quad \frac{G'}{G} = \frac{F_{xx} + F_{yy}}{F} = -\lambda \in \mathbb{R}$$

$$\Rightarrow \quad G' = -\lambda G, \qquad F_{xx} + F_{yy} = -\lambda F$$

Assume $F(x,y) = X(x)\,Y(y)$

$$\Rightarrow \quad F_{xx} + F_{yy} = X''Y + XY'' = -\lambda XY$$

$$\Leftrightarrow \quad \frac{X''}{X} = \left(-\frac{Y''}{Y} - \lambda\right) = -k$$

$$\Rightarrow \quad X'' + kX = 0$$

$$Y'' + (\lambda - k)Y = 0$$

(b) Given,

$$F''(x) + \lambda F(x) = 0, \quad \lambda \in [0, \infty).$$

The following table contains different boundary conditions for the ODE. Fill in each table element with either a yes or a no.

	Boundary value problem has a cosine solution	Boundary value problem has a sine solution	Boundary value problem has a nontrivial constant solution
$F'(0) = 0,\ F'(L) = 0$	✓	✗	✓
$F(0) = 0,\ F'(L) = 0$	✗	✓	✗
$F(0) = 0,\ F(L) = 0$	✓	✗	✗
$F'(0) = 0,\ F(L) = 0$	✗	✓	✗

3. (10 Points) Show that $u(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$ is a solution to $u_{xx} + u_{yy} + u_{zz} = 0$.

$$u_x = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x$$

$$\Rightarrow u_{xx} = -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2}$$

$$\Rightarrow u_{xx}+u_{yy}+u_{zz} = \frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} + \frac{-3}{(x^2+y^2+z^2)^{3/2}} =$$

$$= \frac{3-3}{(x^2+y^2+z^2)} = 0 \checkmark$$

4. (10 Points) Given

$$F'' + \lambda F = 0, \quad \lambda \in [0, \infty). \tag{1}$$

Find all nontrivial solutions F and constants λ that satisfy the boundary conditions:

(a) $\underbrace{F'(0) = 0}$ and $F'(L) = 0$

$\Rightarrow$ Cosine $\quad F(x) = \cos(\sqrt{\lambda}x) \Rightarrow F'(L) = -\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0$

$\Rightarrow \sqrt{\lambda_n} = \frac{n\pi}{L}, \; n = 1, 2, 3$

$\Rightarrow F_n(x) = \cos(\sqrt{\lambda_n}x)$

Note:

$F(x) = \alpha \in \mathbb{R}$

also satisfies ~~when~~ BVP when $\lambda = 0$

(b) $\underbrace{F'(0) = 0}$ and $F(L) = 0$

$\Rightarrow$ Cosine $\Rightarrow F(L) = \cos(\sqrt{\lambda}L) = 0 \Rightarrow \sqrt{\lambda_n} = \text{odd}\,\frac{\pi}{2L} =$

$= (2n-1)\frac{\pi}{2L}, \; n = 1, 2, 3, \ldots$

$\Rightarrow F_n(x) = \cos(\sqrt{\lambda_n}x)$

5. (10 Points) Solve the following PDE.

$$\frac{\partial^2 u}{\partial^2 t} = \frac{\partial^2 u}{\partial x^2}, \tag{2}$$
$$u(0,t) = 0, \qquad u(\pi, t) = 0, \tag{3}$$
$$u(x,0) = f(x), \tag{4}$$
$$u_t(x,0) = g(x) \tag{5}$$

Step 1: $u(x,t) = F(x)G(t)$

$(2) \Leftrightarrow \quad \frac{G''}{G} = \frac{F''}{F} = -\lambda \in \mathbb{R} \quad (*)$

Step 2: $(3) \Rightarrow \quad F(0) = 0, \quad F(L) = 0$

$\Rightarrow \quad F_n(x) = \sin(\sqrt{\lambda_n} x), \quad \sqrt{\lambda_n} = \frac{n\pi}{L}, \; n = 1, 2, 3, \ldots$

$\lambda_n + (*) \Rightarrow \quad G_n(t) = c_n \cos(\sqrt{\lambda_n} t) + d_n \sin(\sqrt{\lambda_n} t)$

Step 3: $u(x,t) = \sum_{n=1}^{\infty} \sin(\sqrt{\lambda_n} x)\left[c_n \cos(\sqrt{\lambda_n} t) + d_n \sin(\sqrt{\lambda_n} t)\right]$

$(4) \Rightarrow \quad f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin(\sqrt{\lambda_n} x) \Rightarrow c_n = \frac{2}{L} \int_0^L f(x) \sin(\sqrt{\lambda_n}$

$L = \pi$

$(5) \quad g(x) = u_t(x,0) = \sum_{n=1}^{\infty} d_n \sqrt{\lambda_n} \sin(\sqrt{\lambda_n} x)$

$\Rightarrow \quad d_n = \frac{2}{L \cdot \sqrt{\lambda_n}} \int_0^L g(x) \sin(\sqrt{\lambda_n} x)\, dx$