

MATH 348 - SPRING 2008

HOMEWORK 8

1. CONSIDER THE 1-D WAVE EQUATION

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (1)$$

$$x \in (0, L) \quad t \in (0, \infty) \quad c^2 = \frac{1}{\rho} \quad (2)$$

$$f(x) = \begin{cases} x & 0 \leq x \leq L \\ -x+2L & L \leq x \leq 2L \end{cases} \quad (3)$$

$$u_x(0, t) = 0 \quad u(2L, t) = 0 \quad (4)$$

$$u(x, 0) = f(x) \quad u_t(x, 0) = g(x) \quad (5)$$

(a) ASSUME THAT $u(x, t) = F(x)G(t)$ AND USE SEPARATION OF VARIABLES TO FIND THE GENERAL SOLUTION TO (1)-(2) THAT SATISFIES (4)-(5).

$$u(x, t) = F(x)G(t) \quad \frac{\partial^2 u}{\partial t^2} = F(x)G''(t) \quad \frac{\partial^2 u}{\partial x^2} = F''(x)G(t)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow F(x)G''(t) = c^2 F''(x)G(t)$$

$$\Rightarrow \frac{F''(x)}{F(x)} = \frac{G''(t)}{c^2 G(t)} = -K$$

$$F''(x) + KF(x) = 0 \quad F'(0) = 0 \quad F(2L) = 0$$

$$K > 0$$

$$F(x) = C_1 \cos(\sqrt{K}x) + C_2 \sin(\sqrt{K}x)$$

$$F'(0) = C_2 \sqrt{K} = 0$$

$$F(2L) = C_1 \cos(\sqrt{K}2L) = 0 \Rightarrow \sqrt{K} = \frac{(2n-1)\pi}{4L} \quad n = 1, 2, 3, \dots$$

$$F(x) = \sum_{n=1}^{\infty} C_n \cos\left(\frac{(2n-1)\pi}{4L}x\right)$$

$$G''(t) + K^2 G(t) = 0 \quad K > 0$$

$$G(t) = B_n \cos(C\sqrt{K}t) + B'_n \sin(C\sqrt{K}t)$$

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$$u(x,t) = f(x)g(t)$$

$$= \sum_{n=1}^{\infty} (B_n \cos(\sqrt{k}t) + B_n^* \sin(\sqrt{k}t)) \cos\left(\frac{(2n-1)\pi}{4L}x\right)$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} \cos\left(\frac{(2n-1)\pi}{4L}x\right) B_n$$

$$\Rightarrow B_n = \frac{1}{L} \int_0^{2L} f(x) \cos\left(\frac{(2n-1)\pi}{4L}x\right) dx$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \cos\left(\frac{(2n-1)\pi}{4L}x\right) B_n^*$$

$$\Rightarrow B_n^* = \frac{1}{C\sqrt{k}L} \int_0^{2L} g(x) \cos\left(\frac{(2n-1)\pi}{4L}x\right) dx$$

$$u(x,t) = \sum_{n=1}^{\infty} (B_n \cos(\sqrt{k}t) + B_n^* \sin(\sqrt{k}t)) \cos\left(\frac{(2n-1)\pi}{4L}x\right)$$

where $B_n = \frac{1}{L} \int_0^{2L} f(x) \cos\left(\frac{(2n-1)\pi}{4L}x\right) dx$

$$B_n^* = \frac{1}{C\sqrt{k}L} \int_0^{2L} g(x) \cos\left(\frac{(2n-1)\pi}{4L}x\right) dx$$



(b) SOLVE FOR THE UNKNOWN CONSTANTS ASSUMING (3)
AND ZERO INITIAL VELOCITY.

$$f(x) = \begin{cases} x & 0 < x \leq L \\ -x+2L & L < x \leq 2L \end{cases}$$

$$B_n = \frac{1}{L} \int_0^{2L} f(x) \cos\left(\frac{(2n-1)\pi}{4L}x\right) dx$$

$$= \frac{1}{BL} \left[\int_0^L x \cos\left(\frac{(2n-1)\pi}{4L}x\right) dx + \int_L^{2L} (-x+2L) \cos\left(\frac{(2n-1)\pi}{4L}x\right) dx \right]$$

$$= \frac{16L}{(2n-1)^2\pi^2} \left[2 \cos\left(\frac{(2n-1)\pi}{4}\right) - 1 \right]$$

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$$g(x) = 0 \leftarrow \text{ZERO INITIAL VELOCITY}$$

$$B_0^* = \frac{1}{c\sqrt{\rho}L} \int_0^{2L} g(x) \cos\left(\frac{(2n-1)\pi}{4L}x\right) dx = 0$$

$$B_n^* = \frac{16L}{(2n-1)^2\pi^2} \left[2\cos\left(\frac{(2n-1)\pi}{4}\right) - 1 \right]$$

$$B_n^* = 0$$

2. CONSIDER THE 1-D WAVE EQUATION WITH BOUNDARY CONDITIONS:

$$u_x(0, t) = 0 \quad u_x(2L, t) = 0 \quad (6)$$

AND INITIAL CONDITIONS:

$$u(x, 0) = f(x) \quad u_t(x, 0) = g(x) \quad (7)$$

- (a) FIND THE GENERAL SOLUTION TO (1)-(2) WHICH SATISFIES (6)-(7).

$$u(x, t) = F(x)G(t) \quad \frac{\partial^2 u}{\partial t^2} = F(x)G''(t) \quad \frac{\partial^2 u}{\partial x^2} = F''(x)G(t)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow F(x)G''(t) = c^2 F''(x)G(t)$$

$$\frac{F''(x)}{F(x)} = \frac{G''(t)}{c^2 G(t)} = -K$$

$$F''(x) + K F(x) = 0 \quad F'(0) = F'(2L) = 0$$

$$K = 0$$

$$F(x) = C_1 + C_2 x$$

$$F'(0) = C_2 = 0$$

$$F'(2L) = C_2 = 0$$

$$F(x) = C_1$$

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 $K > 0$

$$F(x) = C_1 \cos(\sqrt{K}x) + C_2 \sin(\sqrt{K}x)$$

$$F'(0) = C_2 \sqrt{K} = 0 \Rightarrow C_2 = 0$$

$$F'(zL) = C_1 \sqrt{K} \sin(\sqrt{K} zL) = 0 \Rightarrow \sqrt{K} = \frac{n\pi}{zL}$$

$$F(x) = C_1 \cos\left(\frac{n\pi}{zL}x\right) \quad n=1, 2, 3, \dots$$

$$G''(t) + C^2 K G(t) = 0$$

 $K = 0$

$$G(t) = C_1 + C_2 t$$

 $K > 0$

$$G(t) = B_0 \cos(c\sqrt{K}t) + B_n^* \sin(c\sqrt{K}t)$$

$$u(x, t) = F(x)G(t) = A_0 + B_0 t + \sum_{n=1}^{\infty} (B_n \cos(c\sqrt{K}t) + B_n^* \sin(c\sqrt{K}t)) \cos\left(\frac{n\pi}{zL}x\right)$$

$$u(x, 0) = f(x) = A_0 + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi}{zL}x\right)$$

$$A_0 = \frac{1}{zL} \int_0^{zL} f(x) dx \quad B_n = \frac{1}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{zL}x\right) dx$$

$$u_n(x, 0) = g(x) = B_0 + \sum_{n=1}^{\infty} B_n^* c\sqrt{K} \cos\left(\frac{n\pi}{zL}x\right)$$

$$B_0 = \frac{1}{zL} \int_0^{zL} g(x) dx \quad B_n^* = \frac{1}{c\sqrt{KL}} \int_0^L g(x) \cos\left(\frac{n\pi}{zL}x\right) dx$$

$$u(x, t) = A_0 + B_0 t + \sum_{n=1}^{\infty} [B_n \cos(c\sqrt{K}t) + B_n^* \sin(c\sqrt{K}t)] \cos\left(\frac{n\pi}{zL}x\right)$$

$$A_0 = \frac{1}{zL} \int_0^{zL} f(x) dx \quad B_n = \frac{1}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{zL}x\right) dx$$

$$B_0 = \frac{1}{zL} \int_0^{zL} g(x) dx \quad B_n^* = \frac{1}{c\sqrt{KL}} \int_0^L g(x) \cos\left(\frac{n\pi}{zL}x\right) dx$$

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- (b) LET $L=1$ AND SOLVE FOR THE UNKNOWN CONSTANTS ASSUMING (3) AND ZERO INITIAL VELOCITY.

$$f(x) = \begin{cases} x & 0 < x \leq 1 \\ -x+2 & 1 < x < 2 \end{cases}$$

$$A_0 = \frac{1}{2L} \int_0^{2L} f(x) dx = \frac{1}{2} \left[\int_0^1 x dx + \int_1^2 (-x+2) dx \right] = \frac{1}{2}$$

$$B_n = \frac{1}{L} \int_0^{2L} f(x) \cos\left(\frac{n\pi}{2L}x\right) dx$$

$$= \int_0^1 x \cos\left(\frac{n\pi}{2L}x\right) dx + \int_1^2 (-x+2) \cos\left(\frac{n\pi}{2L}x\right) dx$$

$$= \frac{4}{n^2\pi^2} \left[2 \cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1 \right]$$

$$g(x) = 0 \quad \leftarrow \text{ZERO INITIAL VELOCITY}$$

$$B_0 = \frac{1}{2L} \int_0^{2L} g(x) dx = 0$$

~~$$B_n^* = \frac{1}{c\sqrt{KL}} \int_0^{2L} g(x) \cos\left(\frac{n\pi}{2L}x\right) dx = 0$$~~

$$A_0 = \frac{1}{2}$$

$$B_0^* = 0$$

$$B_n = \frac{4}{n^2\pi^2} [2 \cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1]$$

$$B_n^* = 0$$

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- [3]** ASSUME THAT $u(x,t) = Ae^{i(kx-\omega t)}$ IS A SOLUTION TO THE FOLLOWING WAVE-LIKE EQUATION:

$$u_{tt} - u_{xx} + u = 0 \quad (*)$$

SHOW THAT THE PHASE VELOCITY $c_p = \frac{\omega}{k} = \pm \sqrt{1+k^2}$

$$u_{tt} = Aw^2e^{i(kx-\omega t)}$$

$$u_{xx} = Ak^2e^{i(kx-\omega t)}$$

$$u_{tt} - u_{xx} + u = Aw^2e^{i(kx-\omega t)} - Ak^2e^{i(kx-\omega t)} + Ae^{i(kx-\omega t)} = 0$$

$$= w^2 - k^2 + 1 = 0$$

$$= \frac{w^2}{k^2} - 1 + \frac{1}{k^2} = 0$$

$$c_p = \frac{w}{k} = \sqrt{1-k^2}$$

- [4]** SHOW THAT $u(x,t)$ IS A SOLUTION TO THE ONE-DIMENSIONAL WAVE EQUATIONS.

$$u(x,t) = \frac{1}{2}[u_o(x-ct) + u_o(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} v_o(y) dy$$

ASSUME THAT $v_o(y)$ HAS AN ANTIDERIVATIVE, $g(y)$;

THEN $\frac{1}{2c} \int_{x-ct}^{x+ct} v_o(y) dy$ BECOMES $\frac{1}{2c} [g(x+ct) - g(x-ct)]$

$$u_{tt} = \frac{1}{2} [c^2 u''_o(x-ct) + c^2 u''_o(x+ct)] + \frac{1}{2c} [c^2 g''(x+ct) - c^2 g''(x-ct)]$$

$$u_{xx} = \frac{1}{2} [u''_o(x-ct) + u''_o(x+ct)] + \frac{1}{2c} [g''(x+ct) - g''(x-ct)]$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} = \frac{c^2}{2} [u''_o(x-ct) + u''_o(x+ct)] + \frac{c^2}{2} [g''(x+ct) - g''(x-ct)]$$

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2. CONSIDER THE NON-HOMOGENEOUS 1-D WAVE EQUATION

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + F(x, t) \quad (10)$$

LETTING $F(x, t) = A \sin(\omega t)$ GIVES THE FOLLOWING FOURIER SERIES FOR F

$$F(x, t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi x}{L}\right) \quad (14)$$

$$f_n(t) = \frac{2A}{n\pi} (1 - (-1)^n) \sin(\omega t) \quad (15)$$

(a) SHOW THAT SUBSTITUTING (14)-(15) INTO (10) GIVES

$$G_n'' + \left(\frac{c n \pi}{L}\right)^2 G_n = \frac{2A}{n\pi} (1 - (-1)^n) \sin(\omega t) \quad (16)$$

$$F(x, t) = F_n(x) f_n(t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow F_n(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)$$

$$u(x, t) = F_n(x) G_n(t)$$

$$\frac{\partial^2 u}{\partial t^2} = F_n(x) G_n'(t) = \sum_{n=1}^{\infty} G_n'(t) \sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = F_n''(x) G_n(t) = \sum_{n=1}^{\infty} -\left(\frac{c n \pi}{L}\right)^2 G_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + F(x, t)$$

$$\Rightarrow \sum_{n=1}^{\infty} G_n'' \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} -\left(\frac{c n \pi}{L}\right)^2 G_n \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \sum_{n=1}^{\infty} G_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} \left[-\left(\frac{c n \pi}{L}\right)^2 G_n + f_n(t) \right] \sin\left(\frac{n\pi x}{L}\right) \leftarrow$$

$$\Rightarrow G_n' = -\left(\frac{c n \pi}{L}\right)^2 G_n + f_n(t)$$

$$\Rightarrow G_n'' + \left(\frac{c n \pi}{L}\right)^2 G_n = \frac{2A}{n\pi} (1 - (-1)^n) \sin(\omega t)$$

FOR THIS STATEMENT
TO BE TRUE,
THE COEFFICIENTS
MUST BE EQUAL

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(b) THE SOLUTION TO (16) IS GIVEN BY

$$G(t) = B_n \cos\left(\frac{cn\pi}{L}t\right) + B_n^* \sin\left(\frac{cn\pi}{L}t\right) + G_p(t)$$

- i. IF $w \neq \frac{cn\pi}{L}$, WHAT WOULD BE YOUR CHOICE FOR $G_p(t)$
IF "YOU WERE USING THE METHOD OF UNDETERMINED
COEFFICIENTS?"

$$G_p(t) = A \cos(wt) + B \sin(wt)$$

- ii. IF $w = \frac{cn\pi}{L}$, WHAT WOULD BE YOUR CHOICE FOR $G_p(t)$?

$$G_p(t) = At \cos\left(\frac{cn\pi}{L}t\right) + Bt \sin\left(\frac{cn\pi}{L}t\right)$$

- iii. FOR (ii), WHAT IS THE $\lim_{t \rightarrow \infty} u(x, t)$?

$$\lim_{t \rightarrow \infty} u(x, t) = \infty$$

- iv. WHAT DOES THIS LIMIT IMPLY PHYSICALLY?

THIS IS CALLED RESONANCE AND IMPLIES THAT THE MAGNITUDE OF OSCILLATION APPROACHES INFINITY; OR THAT THE OSCILLATING OBJECT BREAKS.