Quote of Lecture 12
Jenny said when she was just five years old there was nothin' happenin' at all. Every time she puts on a radio there was nothin' goin' down at all, not at all. Then one fine mornin' she puts on a New York station you know, she don't believe what she heard at all. She started shakin' to that fine fine music you know her life was saved by rock 'n' roll. Despite all the amputations you know you could just go out and dance to the rock 'n' roll station and it was alright.
The Velvet Underground : Rock And Roll (1970)

We are finally at the end of our study of Fourier methods. We have Fourier series to represent periodic functions and Fourier integrals to represent functions, which do not necessarily have a periodic feature. [1] From the Fourier integral one can then derive the so-called complex Fourier transform or just Fourier transform for short.

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx. \quad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega. \tag{1}$$

In the last set of notes we mention that these equations have a striking similarity to complex Fourier series and that statements of energy and symmetry have analogies for Fourier transform. In the following we use our knowledge of Fourier methods to gather insight into physical process that rely on Fourier analysis.

Let's first begin with the following transform pair,

$$\mathfrak{F}\left\{\delta(t)\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(t)e^{-i\omega t}dt = \frac{1}{\sqrt{2\pi}} \iff \mathfrak{F}^{-1}\left\{\frac{1}{\sqrt{2\pi}}\right\} = \delta(t). \tag{2}$$

This statement says if we wish to send a instantaneous pulse of information then its representation in the frequency domain is a constant function. We take this to mean the following,

(1) A completely localized function requires an equal amount/amplitude of **every** possible frequency of oscillation. Since the sum of the squares of these amplitudes is proportional to the energy of the time-signal we conclude that this transmission would require an infinite amount of energy!

This transform highlights a fundamental property of Fourier transforms. That is, if a function is localized in one domain then it is de-localized in the transformed domain. This relationship is the basis for the Heisenberg uncertainty principle of quantum mechanics but also has a place any time the Fourier transform concept is used. The two most important relations are:

- Position-momentum: In physics position and momentum are related by Fourier transform $\Delta x \Delta p \geq \alpha \in \mathbb{R}^+$ and consequently if the position of a quantum particle is highly localized then its momentum is de-localized. Thus if we know exactly where a particle is then we have no idea about where the particle is going.
- Energy-time: In physics and engineering energy and time are related by Fourier transform $\Delta E \Delta t \geq \alpha \in \mathbb{R}^+$ and consequently if the event takes place in an infinitesimal amount of time then it requires an infinite amount of energy.

Though these statements eccentrically highlight important concepts they are motivated with non-rigorous mathematical tricks involving the 'delta-function.' Maybe a more sensible function is given by,

$$f(t) = \begin{cases} A, & -a < t < a, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

[1] Again, we mention that periodicity can be reconstructed via delta 'functions'. Thus making the Fourier integral the more general representation.

This function is commonly called a single finite pulse and can be thought of as a transmission of information A, which takes place for $2a$ units of time. The transform of such a function is,

$$\hat{f}(\omega) = A\sqrt{\frac{2}{\pi}}\frac{\sin(a\omega)}{\omega}, \tag{4}$$

which is commonly called a sinc function or sampling function. From this transform pair one can gather:

- f is localized in time $\implies$ $\hat{f}$ is de-localized in frequency. That is, some amount of almost every frequency is required to construct the single finite pulse of information. Moreover, if $a \to \infty$ then f is a constant function, which we know transforms to a delta function. Thus, if we consider the limit $a \to \infty$ for a sinc function then we ought to get a delta function! [2]
- The most dominant contribution to the representation of f comes from the $\omega = 0$ mode, which is to say f is 'most like' a constant function but requires the presence of other Fourier modes because it isn't a constant function. [3]

So, why is this called a sampling function? That's a good question and is important to shared band-limited frequency communications down an ideal medium. If we assume that f is a signal, which possesses a Fourier transform and that this signal is band-limited in the frequency domain then it is possible using the concept of periodic extension to show that,

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{L}\right)\frac{\sin(tL - n\pi)}{tL - n\pi}, \tag{5}$$

which implies that the original signal can be reconstructed using sinc/sampling functions where the weights of the previous linear combination are given by the original signal sampled every π/L units in time. This result is known as the sampling theorem and from this we conclude:

- The sinc functions are a basis for **all** time signals sent out over a frequency limited communication medium. That is, any signal sent over the radio, telephone or cable line can use the previous procedure for mathematical reconstruction.
- To send signals over these communication channels the signal is not needed at every instantaneous moment in time. That is it needs to be sampled at integer multiples of π/L in time, where L is defined to be the cut-off frequency, for complete loss-less reconstruction.

1. Lecture Goals

Our goals with this material will be:

- Understand the relationship between a function and its Fourier transform as compared to a periodic function and its Fourier coefficients.
- Conceptualize the Fourier transform by applying it to physically motivated systems.

2. Lecture Objectives

The objectives of these lessons will be:

- Calculate the Fourier transform of the delta function and single finite pulse.
- Derive the representation of a signal transmitted via a band-limited frequency channel.

[2] What a weird thing a delta 'function' is. In ODE's you likely considered the limit of rectangles of unit area and now we have the limit of sinc functions. There are, in fact, many more ways to get to a delta 'function'.

[3] A constant function wouldn't truncate for $|t| > a$.