

Lagrangian formulation:

Hamilton's principle

trajectory of a particle b/w time t_1, t_2

is:

$$\delta \int_{t_1}^{t_2} (T-V) dt = 0$$

$$T \rightarrow KE$$

$$V \rightarrow PE$$

δ variation in path.

$$L \equiv T - V = f(x, \dot{x})$$

if forces are conservative, $M = T + V = \text{const.}$

if $\delta \int_{t_1}^{t_2} L dt = 0$ then integral = const.

suppose we are minimizing w.r.t a parameter α

$$\rightarrow \frac{\partial}{\partial \alpha} \int_{t_1}^{t_2} L(x, \dot{x}) dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} \frac{\partial x}{\partial \alpha} + \frac{\partial L}{\partial \dot{x}} \frac{\partial \dot{x}}{\partial \alpha} \right) dt$$

our extremum path is $x(t)$

and our variation is $x(\alpha, t) = x(0, t) + \alpha \eta(t)$

s.t. when $\alpha = 0$ we have the optimum path.

$$\therefore \frac{\partial x}{\partial \alpha} = \eta(t) \quad \frac{\partial \dot{x}}{\partial \alpha} = \frac{\partial \eta}{\partial t} \quad \left. \begin{array}{l} \text{and } \eta(t_1) = \eta(t_2) \end{array} \right\}$$

$$\therefore \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} \eta(t) + \frac{\partial L}{\partial \dot{x}} \frac{\partial \eta}{\partial t} \right) dt$$

integrate second term by parts:

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{x}} \frac{\partial \eta}{\partial t} dt = \frac{\partial L}{\partial \dot{x}} \eta(t) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) \eta(t) dt$$

$$u = \frac{\partial L}{\partial \dot{x}} \quad du = \frac{d}{dt}$$

$$dv = \frac{\partial \eta}{\partial t} \quad v = \eta$$



together:

$$\int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \eta(t) dt = \text{const.}$$

since $\eta(t)$ is an arbitrary function, $\text{const} \rightarrow 0$
then

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} \quad \text{Lagrangian eq. of motion.}$$

Field Lagrangian

HM 4.9

$$\mathcal{L} \equiv T - U$$

for charge in static field (electrostatic)

$$T = \frac{1}{2} m v^2$$

$$U = q\phi$$

$$\left. \begin{array}{l} T = \frac{1}{2} m v^2 \\ U = q\phi \end{array} \right\} \mathcal{L} = \frac{1}{2} m v^2 - q\phi$$

what if magnetic field is present?

answer:

$$\rightarrow \mathcal{L} = \frac{1}{2} m v^2 + \frac{q}{c} \vec{v} \cdot \vec{A} - q\phi$$

This works if we get the same eqns of motion. (ie. Lorentz force eqn)

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial v_i} = \frac{\partial \mathcal{L}}{\partial x_i}$$

e.g. $\mathcal{L} = \frac{1}{2} m v^2 - mgx$

$$\frac{\partial \mathcal{L}}{\partial v} = m v \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial v} = m \frac{dv}{dt} = ma$$

(momentum)

$$\frac{\partial \mathcal{L}}{\partial x} = -mg = \text{Force}$$

in our case,

$$\frac{\partial \mathcal{L}}{\partial v_i} = m v_i + \frac{q}{c} A_i$$

as a vector

$$\sum \hat{x}_i \frac{\partial \mathcal{L}}{\partial v_i} = \vec{p} + \frac{q}{c} \vec{A}$$

generalized momentum

Now evaluate $\frac{\partial \mathcal{L}}{\partial x_i} = \frac{q}{c} \frac{\partial}{\partial x_i} (\vec{u} \cdot \vec{A}) - q \frac{\partial \Phi}{\partial x_i}$

write in vector form:

$$\sum_i \hat{x}_i \frac{\partial \mathcal{L}}{\partial x_i} = \nabla \mathcal{L} = \frac{q}{c} \nabla (\vec{u} \cdot \vec{A}) - q \nabla \Phi$$

use a general vector ID

$$\nabla (\vec{u} \cdot \vec{A}) = (\vec{u} \cdot \nabla) \vec{A} + (\vec{A} \cdot \nabla) \vec{u} + \vec{A} \times (\nabla \times \vec{u}) + \vec{u} \times (\nabla \times \vec{A})$$

here, x, u are indep. variables. \therefore terms 2, 3 $\rightarrow 0$

Note $\nabla \times \vec{A} = \vec{B}$ (term 4)

For term 1:

$$\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + \sum_i \frac{\partial \vec{A}}{\partial x_i} \frac{\partial x_i}{\partial t} = \frac{\partial \vec{A}}{\partial t} + (\vec{u} \cdot \nabla) \vec{A}$$

\equiv "total derivative"

$$\therefore (\vec{u} \cdot \nabla) \vec{A} = \frac{d\vec{A}}{dt} - \frac{\partial \vec{A}}{\partial t}$$

together:

$$\nabla \mathcal{L} = \frac{q}{c} \left(\frac{d\vec{A}}{dt} - \frac{\partial \vec{A}}{\partial t} \right) + \frac{q}{c} \vec{u} \times \vec{B} - q \nabla \Phi$$

eqn of motion is now

$$\sum_i \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial u_i} = \frac{d}{dt} \left(\vec{p} + \frac{q}{c} \vec{A} \right) = \nabla \mathcal{L} = \frac{q}{c} \frac{d\vec{A}}{dt} + q \left(\underbrace{-\nabla \Phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}}_{\vec{E}} \right) + \frac{q}{c} \vec{u} \times \vec{B}$$

dA/dt terms cancel, so

$$\vec{F} = q \vec{E} + \frac{q}{c} \vec{u} \times \vec{B} \quad \checkmark$$

use $\vec{p} \rightarrow \vec{p} + \frac{q}{c} \vec{A}$ for generalized momentum,

and $\mathcal{L} = \frac{1}{2} m u^2 + \frac{q}{c} \vec{u} \cdot \vec{A} - q \Phi$

Extension of Lagrangian to relativistic case.

with no forces ($\vec{E}, \vec{B} \rightarrow 0$)

$$L = -m_0 c^2 \sqrt{1 - \beta^2} = -\frac{m_0 c^2}{\gamma}$$

check:

1st check non-rel limit - $\beta \ll 1$

$$L = -m_0 c^2 \left(1 - \frac{1}{2} \beta^2\right) = \frac{1}{2} m_0 u^2 - \underbrace{m_0 c^2}_{\text{rest mass energy}}$$

2nd check force eqn:

$$\frac{dL}{dx_i} = 0$$

$$\frac{dL}{du_i} = -m_0 c^2 \left(-2u_i/c^2\right) \left(\frac{1}{2}\right) \left(1 - u^2/c^2\right)^{1/2}$$

$$du_i = m_0 u_i \gamma$$

$$\frac{d}{dt} (m_0 u_i \gamma) = 0$$

same eqn as relativistic force.
eqn. 14.39.

relativistic Lagrangian is

$$L = -\frac{m_0 c^2}{\gamma} + \frac{q}{c} \vec{u} \cdot \vec{A} - q\Phi$$

in 4-vector form, $A = (\vec{A}, i\Phi)$

$$U = (\gamma \vec{u}, i\gamma c)$$

$$L = \frac{1}{\gamma} \left(-m_0 c^2 + \frac{q}{c} (U \cdot A)\right)$$