PHGN 462 Homework 10

1) When I teach 361 I try to really hammer home the point that special functions aren't all that special. At least, they don't need to be scary. I think I said something like "Even Bessel functions aren't too bad as long as you stay calm and look up their properties."

Anyway, let's think about a hollow conducting cylindrical waveguide of radius a (this is not the same as the coaxial pair of cylindrical shells that we did/will do in class… just one cylinder). Let the axis of the cylinder be in the $\hat{k}$ direction.

a) There are a lot of different ways to construct solutions to a new geometry. But easiest is to take a look at the general problem that isn't specific to any geometry. Section 14.3 in Pollack and Stump (attached) does this, so read that up until eqn 14.74. That equation should look pretty familiar, because we've done this before: We solved the general problem and then made it a specific problem by applying rectangular boundary conditions. But this time we're going to apply cylindrical boundary conditions, so what I want you to do first is to solve 14.74 using separation of variables in cylindrical coordinates (technically polar since we're in 2D).

b) If you start going after the above with separation of variables, you should end up with a fairly trivial equation for φ and Bessel's equation for γr, where $\gamma^2 = \left(\frac{\omega}{c}\right)^2 - k^2$. Don't panic! Just read about Bessel's equation and the solutions to it (in any resource you prefer). We're not really going to have to do all that much with them. Put everything together and write down the solution for ψ and then for the magnetic field. The solution should be indexed by some integer; let's call it m. Use a complex exponential for the φ equation instead of sines and cosines. Also, don't feel the need to write out the Bessel functions as power series (after all, you never write sines, cosines, or exponentials as power series unless you have a pretty specific need to).

c) Demonstrate how to find the parameter γ corresponding to a particular mode m. With that, you can complete the dispersion relation. Unless you're very clever, you'll probably at some point have to say "And here's where we can't proceed analytically anymore, so numbers." And that's fine. Sometimes that really is the answer.

2) The solutions to the differential equations for V and **A** in the Lorentz gauge (equations 15.5 and 15.6 in Pollack and Stump, or see lecture notes) are pretty clean as long as you evaluate the charge and current densities at the retarded time $t - {}^{r}/_{c}$. This represents the fact that influences from a charge or current travel at some speed c and take an amount of time ${}^{r}/_{c}$ to reach some observation point. None of this is super shocking to most people.

You know what *is* kind of shocking? The retarded potentials for V and **A** (equations 15.19 and 15.20 in Pollack and Stump, or notes) don't only work with $t - {}^{r}/_{c}$. They also work with $t + {}^{r}/_{c}$, called the advanced time. This is an example of the fact that most physical laws are invariant with respect to time reversal – that is, *for the most part*, nature doesn't care which direction time flows; physical laws work just fine in either direction.

What I'd like you to do is to show that the advanced scalar potential, which looks like:

$$V(\vec{x}, t) = \int \frac{\rho\left(\vec{x}', t + \frac{|\vec{x} - \vec{x}'|}{c}\right)}{4\pi\varepsilon_0} \frac{d^3x'}{|\vec{x} - \vec{x}'|}$$

Satisfies the Lorenz-gauge differential equation for V:

$$-\nabla^2 V + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = \frac{\rho}{\varepsilon_0}$$

And I mean for you to do it by direct substitution. Put the V shown into the differential equation shown, do all the derivatives, and show that the equality holds. Don't just repeat the retarded potential derivation from lecture with the sign flipped. And don't just put that equation into Mathematica and have it say that the equality holds. Show me by hand. These derivatives show up a lot in radiation problems, and some practice isn't a bad idea.

Incidentally, the presence of these advanced potentials means that when I say that E&M is intrinsically causal, as I sometimes do, I'm *slightly* fibbing. It's true that our solutions always require some kind of time difference in between cause and effect – it's just that they're not super particular about which one comes *first*.

3) Second peer lecture as described in class

14.3 ■ WAVE GUIDE OF ARBITRARY SHAPE

The rectangular wave guide is most common, but other shapes are possible. In this section we analyze harmonic electromagnetic waves in a wave guide with an arbitrary cross section. The wave guide is infinitely long in the z direction, and the shape of the cross section is independent of z. The boundary surface is a cylinder parallel to the z axis. A cross section of the interior region is bounded by a closed curve C parallel to the xy plane. For simplicity we assume ideal conditions: The exterior of the guide is a perfect conductor and the interior is vacuum. The basic equations for the rectangular wave guide can be generalized to an arbitrary cross section.

TE modes. The fields for a TE wave propagating in the $+z$ direction may be written in the form

$$\mathbf{E}(\mathbf{x}, t) = -\nabla \times \left(\hat{\mathbf{z}}\psi e^{i(kz-\omega t)}\right) = -\left[\nabla \times (\hat{\mathbf{z}}\psi)\right] e^{i(kz-\omega t)} \tag{14.72}$$

$$\mathbf{B}(\mathbf{x}, t) = \frac{k}{\omega}\left[-\nabla\psi + \frac{i\gamma^2}{k}\hat{\mathbf{z}}\psi\right] e^{i(kz-\omega t)}, \tag{14.73}$$

where $\psi(x, y)$ is a scalar function independent of z. These forms are the same as (14.43) and (14.55) used in the analysis of the rectangular wave guide. All four Maxwell equations are satisfied if $\psi(x, y)$ is a solution of the 2D Helmholtz equation

$$\nabla^2\psi = -\gamma^2\psi \tag{14.74}$$

with $\omega^2/c^2 = k^2 + \gamma^2$.[11] All that remains is to impose the boundary conditions.

The normal component of $\mathbf{B}$ must be 0 on C, the boundary curve of a cross section of the guide. What is this condition in terms of $\psi(x, y)$? By (14.73) the normal component of $\mathbf{B}$ at a point on the surface is proportional to $\hat{\mathbf{n}} \cdot \nabla\psi$, where $\hat{\mathbf{n}}$ is the unit normal vector at the point in the plane of C; thus

$$\hat{\mathbf{n}} \cdot \nabla\psi = 0 \quad \text{on} \quad C. \tag{14.75}$$

The tangential components of $\mathbf{E}$ must also be 0 on C, but that requirement leads to the same condition (14.75).

For the TE modes then, $\psi(x, y)$ obeys the 2D Helmholtz equation in the region enclosed by C, with normal derivative 0 on the boundary. Equation (14.74) is an eigenvalue problem, with operator ∇^2 and eigenvalue $-\gamma^2$. That is, $\psi(x, y)$ is the eigenfunction of the 2D Laplacian with the Neumann boundary condition (14.75). There are an infinite number of discrete eigenstates. If the angular frequency ω of the field oscillations is greater than $c\gamma$ then the solution describes a propagating wave, because k is real in that case. But if ω is less than $c\gamma$ then the solution

[11] See Exercise 10.