MATH348 - April 16, 2010
Exam II - 50 Points

NAME: Key

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (10 Points) True/False and Short Response

   (a) Mark each statement as either true or false. No justification is needed.

   i. Suppose f is a periodic function such that $f(-x) = -f(x)$. The Fourier series representation of f will have only cosine functions.

   False, It should have sine fn or be trivial

   ii. The Fourier transform of a function with no symmetry has no symmetry.

   True, See Hw #7 problem 1

   iii. If a periodic function is neither odd nor even then the Fourier series representation will have both sine and cosine functions.

   False, tricky $\pi + \sum_{n=1}^{\infty} b_n \sin(nx)$ has no symmetry.

   iv. Suppose f is non-zero and finite for $x \in (0, L)$ then there exists exactly one Fourier series representation of f.

   False, See 11.3

   v. Real Fourier series are used to represent periodic functions that complex Fourier series cannot.

   False Real F.S. = Complex F.S.

   (b) What is Gibb's phenomenon and when will a Fourier series manifest Gibb's phenomenon?

   Gibb's phenomenon is a highly oscillatory phenomenon associated with a truncated F.S. trying to Represent a jump discontinuity in a periodic fn.

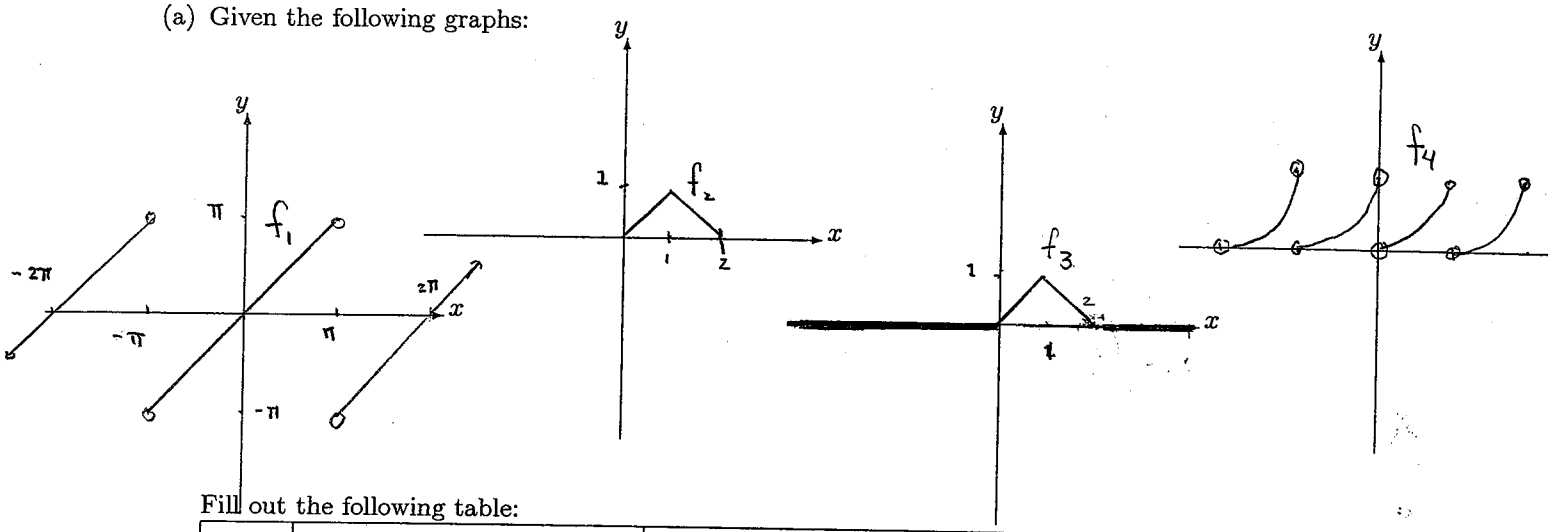
   This will go away ~~where~~ when the F.S. is not truncated with the consequence that the F.S. will average Rtt, Ltt limits of f.

   (c) How is the Fourier integral related to the Fourier series? What is the purpose of each?

   A F.S. Represents a Periodic fn while a F.I. can Represent fn that are nonperiodic. One can get to a F.I. by taking a limit, provided it exists, of a F.S. as $L \to \infty$.

2. (10 Points) Quickies

(a) Given the following graphs:



Fill out the following table:

	Has a Fourier Cosine Series	Has a Fourier Sine Series	Has a Fourier Transform
$f_1(x)$	no	Yes	Yes
$f_2(x)$	Yes	Yes	Yes
$f_3(x)$	no	no	Yes
$f_4(x)$	NO	Yes	Yes

Given the following integrals,

$$\int_{-\pi}^{\pi} f(x)dx = \pi,$$

$$0 = \int_{-\pi}^{\pi} f(x)\cos(nx)dx = \left[\frac{\sin(nx)}{n}\Big|_{-\pi}^{0} + \frac{\sin(nx)}{n}\Big|_{0}^{\pi}\right], \quad \int_{-\pi}^{\pi} f(x)\sin(nx)dx = \left[\frac{\cos(nx)}{n}\Big|_{-\pi}^{0} - \frac{\cos(nx)}{n}\Big|_{0}^{\pi}\right], = \frac{1}{n}\left(2-2(-1)^n\right)$$

$$i\frac{(-1)^n}{n} = \int_{-\pi}^{\pi} g(x)e^{-inx}dx, \quad \cancel{\frac{e^{in\pi} - e^{-in\pi}}{4\pi}} = \int_{-\pi}^{\pi} g(x)dx.$$

0

(b) Fill out the following table:

	Is Even	Is Odd
$f(x)$	no	no
$g(x)$	no	Yes

(c) Find the Fourier transform of $f(x) = \frac{\sqrt{2\pi}}{2i}[\delta(x-2) - \delta(x+2)]$.

$$\mathcal{F}\{f\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\sqrt{2\pi}}{2i}\left[\delta_2 - \delta_{-2}\right] e^{-i\omega x}\,dx =$$

$$= \frac{-2i\sin(2\omega)}{2i} = -\sin(2\omega)$$

(d) What is the real Fourier series representation of the following function,

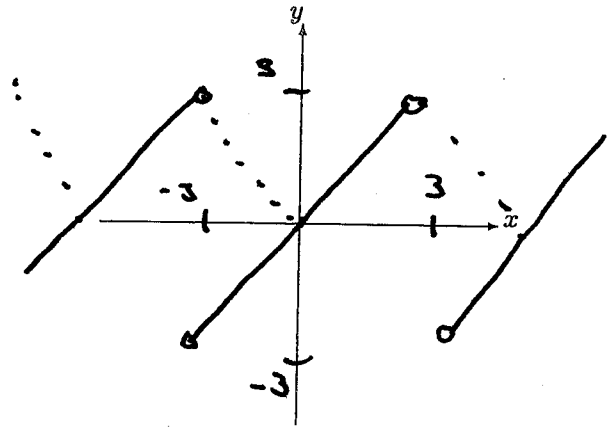
$$f(x) = \pi + 3\cos(2x) - \sin(x).$$

This is the F.S. Rep. of f.

$a_0 = \pi \quad b_1 = -1$

$a_2 = 3 \quad$ all others are zero.

3. (10 Points) Let $f(x) = x$ for $x \in (0, 3)$.



(a) Graph the Fourier cosine half-range expansion of f with a dashed line and the Fourier sine half-range expansion with a solid line on the axes above.

(b) Find the Fourier cosine series half-range expansion of f.

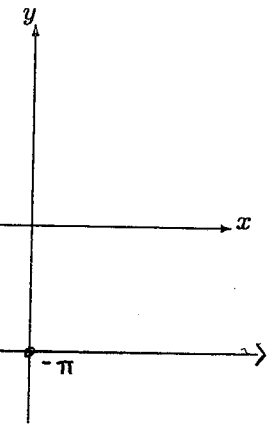
u	dv
x	$\cos\left(\frac{n\pi}{L}x\right)$
1	$\sin\left(\frac{n\pi}{L}x\right)\cdot\frac{L}{n\pi}$
0	$-\cos\left(\frac{n\pi}{L}x\right)\frac{L^2}{n^2\pi^2 L}$

$$a_0 = \frac{1}{L}\int_0^L f\,dx = \frac{1}{3}\cdot\frac{1}{2}\cdot 3\cdot 3 = \frac{3}{2}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx = \frac{2}{L}\left[\frac{L^2}{n^2\pi^2}\left((-1)^n - 1\right)\right] = a_n$$

$$\Rightarrow f_c(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{6}{n^2\pi^2}\left((-1)^n - 1\right)\cos\left(\frac{n\pi}{3}x\right)$$

4. (10 Points) Find the complex Fourier series representation of f given by the graph below.



$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\frac{\pi}{L}x}\,dx =$$

$$= \frac{-\pi}{2L}\left[\frac{-L}{in\pi}\left(e^{-in\frac{\pi}{L}x}\right)\Big|_{-L}^{L}\right] =$$

$$= \frac{1}{2in}\left[e^{-in\pi} - e^{in\pi}\right] =$$

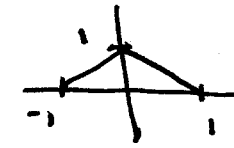
$$= \frac{1}{2in}\left[-2i\sin(n\pi)\right] = 0,\ n \neq 0$$

$$c_0 = \frac{1}{2L}\int_{-L}^{L} -\pi\,dx = -\pi$$

$$\Rightarrow f(x) = -\pi$$

5. (10 Points) Suppose that f is given as,

$$f(x) = \begin{cases} x+1, & -1 \le x < 0 \\ 1-x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}.$$



Calculate the complex Fourier transform of f. Noting the identity, $2\sin^2(y) = 1 - \cos(2y)$, simplify as much as possible.

Note that f is Even

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, d\omega = \frac{2}{\sqrt{2\pi}} \int_0^1 (1-x)\cos(\omega x) \, d\omega =$$

u	dv
$1-x$	$\cos(\omega x)$
-1	$\frac{1}{\omega}\sin(\omega x)$
0	$-\frac{1}{\omega^2}\cos(\omega x)$

$$= \frac{2}{\sqrt{2\pi}} \left[-\left(\frac{\cos(\omega x)}{\omega^2} - \frac{1}{\omega^2} \right) \right] =$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{1 - \cos(\omega)}{\omega^2} \right] =$$

$$= \frac{2}{\sqrt{2\pi}} \frac{2\sin^2(\omega/2)}{\omega^2} = \sqrt{\frac{2}{\pi}} \frac{\sin^2(\omega/2)}{\omega^2} =$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\sin^2(\omega/2)}{\left(\frac{\omega}{2}\right)^2} = \frac{1}{\sqrt{2\pi}} \operatorname{sinc}^2(\omega/2)$$