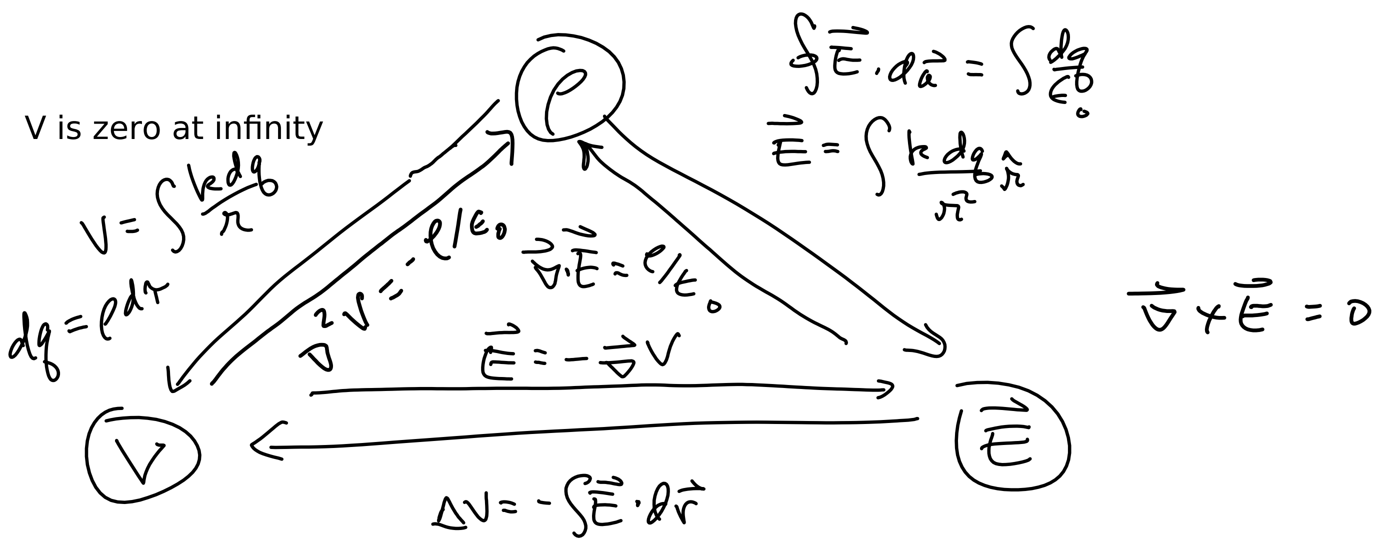
I will not be here on Friday due to a conference. Friday's lecture notes will be posted after Wed. class. There will be InkSurvey exercises you will do as part of this lecture, and due Monday at 9.

Lecture 17 Shadowitz: 5.1 the vector potential

Here is our model of electrostatics ρ does not depend on time

V is zero at infinity



$$\vec{F} = q\vec{E} \qquad W_{nc} = \Delta(KE + PE)$$

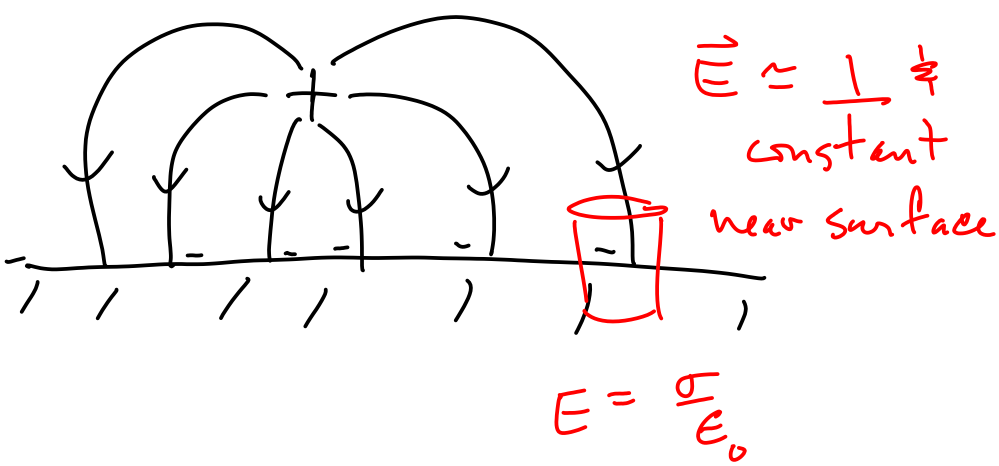
$$W_{me} = \int V\,dq = \Delta PE \quad \text{or} = \frac{1}{2}\int V\,dq$$

To assemble a charge distribution

$$W = \frac{\epsilon_0}{2}\left[\int E^2 d\tau + \oint V\vec{E}\cdot d\vec{a}\right]$$

Is this energy stored in the charge distribution or electric fields?

Hmwk problem review:



Unique solution is given by $\boxed{\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0} + \boxed{\vec{\nabla}\times\vec{E} = 0}$

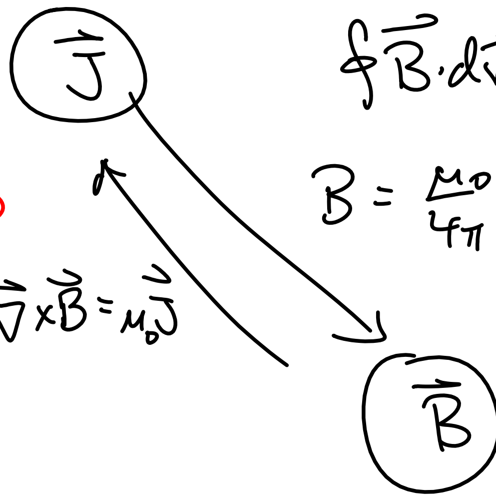
$+$ $\boxed{\text{Boundary conditions}}$ **Gauss's and Ampere's laws are used to determine the boundary conditions**

Here is our model of magnetostatics:

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t} = 0$$

$$\oint \vec{J}\cdot d\vec{a} = -\frac{\partial}{\partial t}\int \rho\, d\tau = 0$$

J does not depend on time



$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

$$\oint \vec{B}\cdot d\vec{r} = \mu_0 \underbrace{\int \vec{J}\cdot d\vec{a}}_{I_{enclosed}}$$

$$B = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$d\vec{F} = dq\,\vec{v}\times\vec{B} \rightarrow I\,d\vec{r}\times\vec{B} \rightarrow \vec{K}\times\vec{B}\,da \rightarrow \vec{J}\times\vec{B}\,d\tau$$

Questions about the model?

-(creative) How do we introduce a potential for B?

$\vec{\nabla} \cdot \vec{B} = 0$ always

note $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0$ always

How would you prove this?

$\vec{B} = \vec{\nabla} \times \vec{A}$ so $\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \vec{\nabla} \times \vec{A} \equiv 0$

↖ vector potential

Questions?

-(congruous) How do I calculate A?

-(analogy) Is there any analogy between A and V?

-(informational) What advantage is there to using A?

↓ more

$$\vec{E} = -\vec{\nabla} V$$

V is not unique for a unique E.

$$\vec{E} = -\vec{\nabla}(V + \text{constant})$$

Similarly A is not unique for a unique B.

$$\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times(\underbrace{\vec{A}' + \vec{\nabla}\psi}_{\vec{A}}) = \vec{\nabla}\times\vec{A}' + \underbrace{\vec{\nabla}\times\vec{\nabla}\psi}_{\text{always } 0}$$

To uniquely define A we need both the curl and divergence

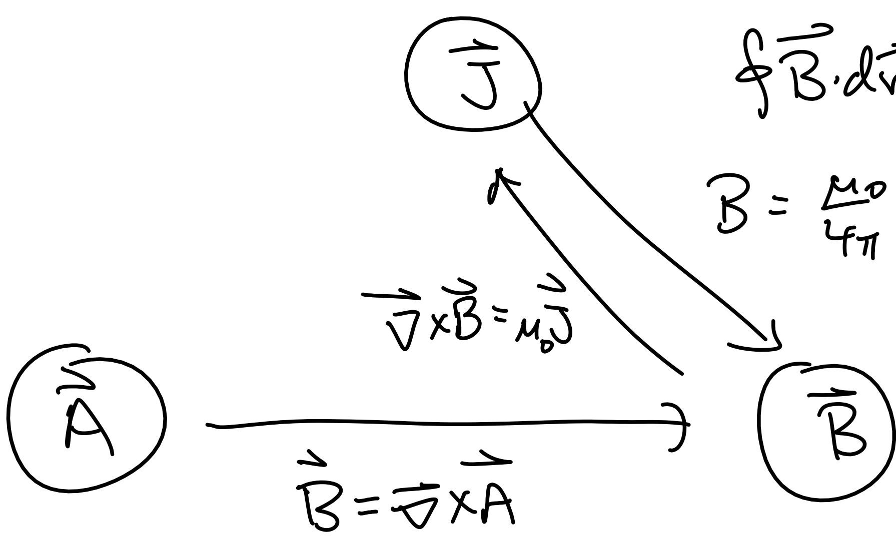
$$\vec{\nabla}\times\vec{A} = \vec{B} \qquad \vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\vec{A}' + \nabla^2\psi$$

The choice for ψ is called the choice of a guage.

For magneto-statics we choose ψ such that $\boxed{\vec{\nabla}\cdot\vec{A} = 0}$

Coulomb guage

Here is our model of magnetostatics:



$$\oint \vec{B}\cdot d\vec{r} = \mu_0 \underbrace{\int \vec{J}\cdot d\vec{a}}_{I_{\text{enclosed}}}$$

$$B = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\cdot\vec{A} = 0$$

Let's go counterclockwise to see if we can find J given A.

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\underbrace{\vec{\nabla} \times \vec{\nabla} \times \vec{A}} = \mu_0 \vec{J}$$

Vector identity

$$\overbrace{\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}} = \mu_0 \vec{J}$$

$\vec{\nabla} \cdot \vec{A} \to 0$ Coulomb guage choice

vector Laplacian

$$\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J} \longrightarrow \hat{x}\, \nabla^2 A_x + \hat{y}\, \nabla^2 A_y + \hat{z}\, \nabla^2 A_z = -\mu_0 \vec{J}$$

$$\nabla^2 A_x = -\mu_0 J_x$$

$$A_x = \int \frac{J}{r}\, dx'dy'dz$$

scalar Laplacian

$$\nabla^2 V = -\rho/\epsilon_0$$

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(x',y',z')}{r}\, dx'dy'dz'$$

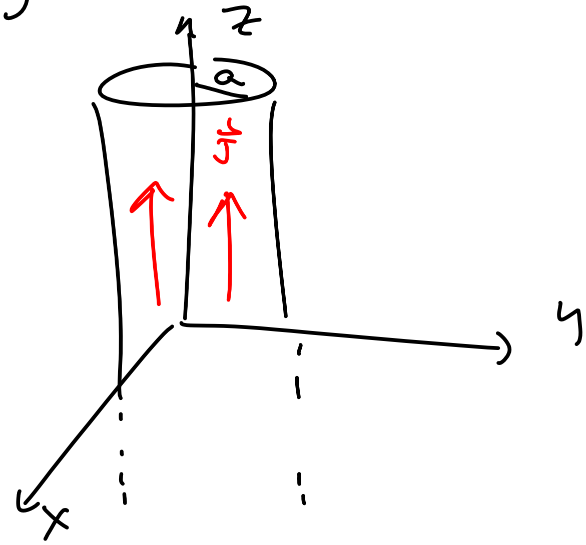
assuming V goes to zero at infinity

By analogy

$$\begin{cases} A_x(\vec{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \dfrac{J_x(\vec{r}\,')\,d\tau'}{|\vec{r}-\vec{r}\,'|} \\ A_y = \displaystyle\int J_y \\ A_z = \displaystyle\int J_z \end{cases}$$

assuming J_x goes to zero at infinity

Example:



$$\vec{B} = \frac{\mu_0 I_0}{2\pi r}\hat{\varphi} = \vec{\nabla}\times\left[A_\varphi\hat{\varphi} + A_r\hat{r} + A_z\hat{z}\right]$$

We know $\vec{A}$ points in the direction of $\vec{J}$ so $A_\varphi = A_r = 0$

$$\vec{B} = \left[\frac{1}{r}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right]\hat{r} + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\hat{\varphi} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\varphi) - \frac{\partial A_r}{\partial\varphi}\right]\hat{z}$$

$$\vec{B} = \left[\frac{1}{r}\underbrace{\frac{\partial A_z}{\partial\varphi}}_{=0} - \overbrace{\frac{\partial A_\varphi}{\partial z}}^{0}\right]\hat{r} + \left[\overbrace{\frac{\partial A_r}{\partial z}}^{0} - \frac{\partial A_z}{\partial r}\right]\hat{\varphi} + \frac{1}{r}\left[\overbrace{\frac{\partial}{\partial r}(rA_\varphi)}^{0} - \overbrace{\frac{\partial A_r}{\partial\varphi}}^{0}\right]\hat{z}$$

$$\frac{\mu_0 I_0}{2\pi r}\hat{\varphi} = -\frac{dA_z}{dr}\hat{\varphi}$$

$$dA_z = -\frac{\mu_0 I_0}{2\pi}\frac{dr}{r}$$

$$A_z = -\frac{\mu_0 I_0}{2\pi}\ln\left(\frac{r}{a}\right)$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = -\hat{\varphi}\frac{dA_z}{dr} = \frac{\mu_0 I_0}{2\pi}\frac{1}{r}\hat{\varphi}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z} = 0$$

A is zero at infinity