Matrix Inversion, Decomposition and Determinants

Text: 2.2-2.5, 3.1-3.3 Section Overviews: 2.2-2.5, 3.1-3.3

Quote of Homework Four	
The trick is in what one emphasizes. We either make ourselves miserable, or we make ourselves strong. The amount of work is the same.	
	Carlos Castaneda : (1925-1998)

1. Matrix Inversion

Given,

$$\mathbf{A} = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}.$$

1.1. **Matrix Inverse: Take One.** Find $\mathbf{A}^{-1}$ using the Gauss-Jordan method.[1]

1.2. **Matrix Inverse: Take Two.** Find $\mathbf{A}^{-1}$ using the cofactor representation. [2]

1.3. **Solutions to Linear Systems.** Using $\mathbf{A}^{-1}$ find the unique solution to $\mathbf{Ax} = \mathbf{b} = [b_1\, b_2\, b_3]^{\mathrm{T}}$.

1.4. **Left Inversion in Rectangular Cases.** Let $\mathbf{A}^{-1}_{\text{left}} = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}$. Show that $\mathbf{A}^{-1}_{\text{left}}\mathbf{A} = \mathbf{I}$. [3]

1.5. **Right Inversion in Rectangular Cases.** Let $\mathbf{A}^{-1}_{\text{right}} = \mathbf{A}^{\mathrm{T}}(\mathbf{A}\mathbf{A}^{\mathrm{T}})^{-1}$. Show that $\mathbf{A}\mathbf{A}^{-1}_{\text{right}} = \mathbf{I}$.[4]

1.6. **Inversion for Rectangular Matrices.** Let $\mathbf{A}_1 = [2\ 2]^{\mathrm{T}}$ and $\mathbf{A}_2 = [2\ 2]$. Using the previous formula find the left-inverse of $\mathbf{A}_1$ and the right-inverse of $\mathbf{A}_2$. Check your results with the appropriate multiplication.

2. Block Matrix Inversion

Suppose that $\mathbf{A} \in \mathbb{R}^{n\times n}$ can be written in partitioned form as,

$$\mathbf{A} = \begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{bmatrix}. \tag{1}$$

2.1. **Inverse Formula One.** Suppose that $\mathbf{A}$ and $\mathbf{P}$ are non-singular and show that,

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{X} & -\mathbf{P}^{-1}\mathbf{QW} \\ -\mathbf{WRP}^{-1} & \mathbf{W} \end{bmatrix}, \tag{2}$$

where $\mathbf{W} = (\mathbf{S} - \mathbf{RP}^{-1}\mathbf{Q})^{-1}$ and $\mathbf{X} = \mathbf{P}^{-1} + \mathbf{P}^{-1}\mathbf{QWRP}^{-1}$.[5]

[1] The Gauss-Jordan method is another name for row-reduction. For an example see page 124 of the text.

[2] Though row-reduction is more efficient, it is sometimes that case that the whole inverse isn't needed. If particular entries of the inverse matrix are needed then one can use the general inversion formula given by theorem 8 on page 203, which consists of a matrix populated by cofactors.

[3] This matrix is called the left-inverse of $\mathbf{A}$ and it can be shown that if $\mathbf{A} \in \mathbb{R}^{m\times n}$ such that $\mathbf{A}$ has a pivot in every column then the left inverse exists.

[4] This matrix is called the right-inverse of $\mathbf{A}$ and it can be shown that if $\mathbf{A} \in \mathbb{R}^{m\times n}$ such that $\mathbf{A}$ has a pivot in every row then the right inverse exists.

[5] Hint: First, remember that if you are given a candidate for an inverse then you need only check that the appropriate multiplication gives you the identity. Second, you must note that we are working with a matrix whose elements are matrices and when you perform a check you are checking blocks. Thus when you perform the check $\left[\mathbf{AA}^{-1}\right]_{11}$ you are finding the upper-left block of the product matrix and the result should be matrix and not a scalar. What matrix should you get for this block? What about the rest?

2.2. **Inversion Formula Two.** Suppose that $\mathbf{A}$ and $\mathbf{S}$ are non-singular and show that,

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{X} & -\mathbf{XQS}^{-1} \\ -\mathbf{S}^{-1}\mathbf{RX} & \mathbf{W} \end{bmatrix}, \tag{3}$$

where $\mathbf{X} = (\mathbf{P} - \mathbf{QS}^{-1}\mathbf{R})^{-1}$ and $\mathbf{W} = \mathbf{S}^{-1} + \mathbf{S}^{-1}\mathbf{RXQS}^{-1}$.[6]

2.3. **Conclusion.** Show that if $\mathbf{P}, \mathbf{S}, \mathbf{A}$ are all non-singular matrices then $(\mathbf{S} - \mathbf{RP}^{-1}\mathbf{Q})^{-1} = \mathbf{S}^{-1} + \mathbf{S}^{-1}\mathbf{RXQS}^{-1}$.

2.4. **Sanity Check.** Test these formula with $\mathbf{P} = a$, $\mathbf{Q} = b$, $\mathbf{R} = c$, $\mathbf{S} = d$, where $a, b, c, d \in \mathbb{R}$ such that $ad - cb \neq 0$.

3. Invertible Matrix Theory

Assume that $\mathbf{A} \in \mathbb{R}^{n\times n}$ and without using the invertible matrix theorem, prove the following:

3.1. **Spanning Sets.** If $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{A}^{-1}$ exists, then the columns of $\mathbf{A}$ span $\mathbb{R}^n$.

3.2. **Pivot Structure.** If $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{Ax} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^n$, then $\mathbf{A}$ is invertible.

3.3. **Linear Independence.** If the matrix $\mathbf{A}$ is invertible, then the columns of $\mathbf{A}^{-1}$ are linearly independent.

3.4. **Free Variables I.** If the equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, has more than one solution for some $\mathbf{b} \in \mathbb{R}^n$, then the columns of $\mathbf{A}$ do not span $\mathbb{R}^n$.

3.5. **Free Variables II.** If the equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, is inconsistent for some $\mathbf{b} \in \mathbb{R}^n$, then the equation $\mathbf{Ax} = \mathbf{0}$ has a non-trivial solution.

3.6. **Linear Dependence.** If $\mathbf{A}$ is a square matrix with two identical columns then $\mathbf{A}^{-1}$ does not exist.

4. Matrix Decompositions

4.1. **LU Factorization.** Given,

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & -1 & 5 \\ 3 & 7 & -2 & 9 \\ -2 & -3 & 1 & -4 \end{bmatrix}.$$

Determine the LU-Decomposition of the matrix $\mathbf{A}$ **and** check your result for $\mathbf{L}$ by multiplication of three elementary matrices.[7]

4.2. **Spectral Factorization.** Suppose $\mathbf{A} \in \mathbb{R}^{3\times 3}$ admits a factorization $\mathbf{A} = \mathbf{PDP}^{-1}$, where $\mathbf{P} \in \mathbb{R}^{3\times 3}$ is a invertible matrix and $\mathbf{D} \in \mathbb{R}^{3\times 3}$ is the diagonal matrix,[8]

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}. \tag{4}$$

Find a formula for $\lim_{k\to\infty} \mathbf{A}^k$. [9]

4.3. **QR Factorization.** Suppose that $\mathbf{A} = \mathbf{QR}$ where $\mathbf{Q}, \mathbf{R} \in \mathbb{R}^{n\times n}$ are invertible matrices and $\mathbf{R}$ is upper-triangular while $\mathbf{Q}$ is such that $\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}$. Show that for each $\mathbf{b} \in \mathbb{R}^n$ the equation $\mathbf{Ax} = \mathbf{b}$ has a unique solution and without using $\mathbf{R}^{-1}$ find formulas for calculating $\mathbf{x}$.

4.4. **Singular Value Decomposition: Special Case.** Suppose that $\mathbf{A} = \mathbf{U\Sigma V}^{\mathrm{T}}$ where $\mathbf{U}, \mathbf{V} \in \mathbb{R}^{n\times n}$ are invertible with the property that their transposes are their own inverses and $\mathbf{\Sigma}$ is a diagonal matrix with positive entries on the diagonal. Show that $\mathbf{A}$ is an invertible matrix and find a formula for $\mathbf{A}^{-1}$.

5. Determinants

5.1. **Determinants of Inversions.** Show that if $\mathbf{A}$ is invertible, then $\det(\mathbf{A}^{-1}) = \dfrac{1}{\det(\mathbf{A})}$.

5.2. **Determinants of Orthogonal Matrices.** Let $\mathbf{U}$ be a square matrix such that $\mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{I}$. Show that $\det(\mathbf{U}) = \pm 1$.

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[6] Hint: Same as before but now it is easiest to check $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

[7] The matrix $\mathbf{U}$, found by three steps of row reduction on $\mathbf{A}$, will have two pivot columns. These two pivot columns are used to determine the first two columns of $\mathbf{L}_{3\times 3}$. The remaining column of $\mathbf{L}$ is equal the last column of $\mathbf{I}_3$.

[8] A diagonal matrix is a matrix that is both upper <u>and</u> lower triangular. That is $\mathbf{A} \in \mathbb{R}^{m\times n}$ is diagonal if and only if $[\mathbf{A}]_{ij} = 0$ for $i \neq j$.

[9] Hint: First find a formula for $\mathbf{A}^k$ using the spectral factorization. In this formula the exponent should only change the $\mathbf{D}$ matrix.

5.3. **Determinants of Similar Matrices.** Let $\mathbf{A}$ and $\mathbf{P}$ be square matrices such that $\mathbf{P}^{-1}$ exists. Show that $\det(\mathbf{PAP}^{-1}) = \det(\mathbf{A})$.

5.4. **Row-Operation Sanity Check.** Given the following for matrices:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}.$$

Calculate the determinants of the previous matrices by theorem 2.2.4. In each case, state the row-operation used on $\mathbf{A}$ to get to $\mathbf{B}, \mathbf{C}, \mathbf{D}$ and describe how it affects the determinant.

5.5. **Scaling Properties.** Find a formula for $\det(r\mathbf{A})$ where $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $r \in \mathbb{R}$.

5.6. **Vandermonde Matrix.** Given,

$$\mathbf{A} = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}.$$

5.7. ***Vandermonde Determinant.*** Show that the $\det(\mathbf{A}) = (c-a)(c-b)(b-a)$.[10]

5.8. **Multi-linearity.** The determinant is not, in general, a linear mapping. That is, det: $\mathbb{R}^{n\times n} \to \mathbb{R}$ is **not**, in general, such that, $\det(\mathbf{A}+\mathbf{B}) = \det(\mathbf{A})+\det(\mathbf{B})$. The determinant is, in general, *multilinear*.[11] Show this for the domain $\mathbb{R}^{3\times 3}$ by verifying that $\det(\mathbf{A}) = \det(\mathbf{B}) + \det(\mathbf{C})$, where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are given as,[12]

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & u_1+v_1 \\ a_{21} & a_{22} & u_2+v_2 \\ a_{31} & a_{32} & u_3+v_3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & u_1 \\ a_{21} & a_{22} & u_2 \\ a_{31} & a_{32} & u_3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} a_{11} & a_{12} & v_1 \\ a_{21} & a_{22} & v_2 \\ a_{31} & a_{32} & v_3 \end{bmatrix}.$$

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[10] Hint: It would be in your best interest to use row-reduction methods. This, of course, generalizes. `http://en.wikipedia.org/wiki/Vandermonde_matrix`

[11] A multilinear map is a mathematical function of several vector variables that is linear in each variable. That is, if all columns except one are fixed, then the determinant is a linear function of that one column. See `http://en.wikipedia.org/wiki/Multilinear_map` for more information.

[12] The easiest way to do this is by considering a cofactor expansion down the third column of $\mathbf{A}$. In this case the sums will appear as prefactors and distribution of multiplication over addition breaks the expansion into two expansions.